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In a recent paper (see [11]), L. Greengard and V. Rokhlin introduce a numerical technique for the rapid solution of integral equations resulting from linear two-point boundary value problems for second order ordinary differential equations. In this paper, we extend the method to systems of ordinary differential equations. After reducing the system of differential equations to a system of second kind integral equations, we discretize the latter via a high order Nyström scheme. A somewhat involved analytical apparatus is then constructed which allows for the solution of the discrete system using $O(N \cdot p^2 \cdot n^3)$ operations, with N the number of nodes on the interval, p the desired order of convergence, and n the number of equations in the system. Thus, the advantages of the integral equation formulation (small condition number, insensitivity to boundary layers, insensitivity to end-point singularities, etc.) are retained, while achieving a computational efficiency previously available only to finite difference or finite element methods.

We in addition present a Newton method for solving boundary value problems for nonlinear first order systems in which each Newton iterate is the solution of a second kind integral equation; the analytical and numerical advantages of integral equations are thus obtained for nonlinear boundary value problems.

On the Numerical Solution of Two-Point Boundary Value Problems II

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Research Report YALEU/DCS/RR-802
June 1990

The authors were supported by the Office of Naval Research under Grant N00014-89-J-1527.
Approved for public release: distribution is unlimited.

Keywords: *Two-Point Boundary Value Problems, Integral Equations, Chebyshev Polynomials, Approximation Theory*

I. Introduction

Second kind integral equations have been a popular analytical tool in the study of ordinary differential equations for nearly a century. When boundary value problems are being considered, the integral equations which arise are of the Fredholm type. From an abstract viewpoint, the advantage of this formulation is that many properties of the solution are readily apparent; from a computational viewpoint, the linear systems which arise from discretization are generally well-conditioned. An ill-behaved differential equation can often be reduced to a perfectly tractable integral equation by means of an appropriate choice of the "background" Green's function. Standard finite difference and finite element methods, on the other hand, which discretize the original differential equation, encounter serious numerical difficulties when the solution possesses derivatives of large magnitude (boundary layers). A second advantage is that there exist extremely stable, high order numerical methods for the solution of second kind Fredholm equations, while the order of convergence of most practical schemes for the solution of ordinary differential equations tends to be limited, even if Richardson extrapolation and deferred correction approaches are considered.

Despite all these advantages, integral equations are virtually never used as a numerical tool for the solution of two-point boundary value problems, since their discretization leads to dense systems of linear algebraic equations, and the solution of a dense linear system of dimension N requires order $O(N^3)$ arithmetic operations. Finite difference and finite element schemes lead to banded systems of linear algebraic equations, and the solution of the latter requires order $O(N)$ arithmetic operations, with N the dimension of the problem. This makes the use of integral equations extremely unattractive as a numerical tool, despite their superior analytical properties. A similar difficulty is encountered when spectral methods are applied to boundary value problems. They yield high order accuracy, but result in dense systems of linear algebraic equations.

Recently, [11] presented a fast numerical algorithm for solving two-point boundary value problems for second order differential equations. By solving the problems as second kind integral equations, one obtains the superior properties of integral equations over differential equations. By using the technique of [11], integral equations arising from boundary value problems are solved in order O(N) arithmetic operations.

In this paper, we extend the results of [11] by showing that integral equations arising from two-point boundary value problems for systems of ordinary differential equations can also be solved for a cost proportional to the number of nodes N. We in addition present a Newton method for solving boundary value problems for nonlinear first order systems in which each Newton iterate is the solution of a second kind integral equation.

The plan of this paper is as follows: in Section 2 we summarize both the theory of Green's functions for first order linear systems and the theory of Newton methods for first order nonlinear systems, in Section 3 we develop the analytical apparatus to be used, and in Section 4 we describe the numerical schemes themselves. The performance of the methods is illustrated in Section 5 with numerical examples. Our conclusions are discussed in Section 6.

The present paper is similar to [11] in that while it is based on a sequence of fairly simple observations, the details of the algorithm are somewhat involved. We attempt in this paper to

present both cursory, qualitative descriptions as well as detailed, rigorous proofs.

II. Mathematical Preliminaries

In this section, we summarize the relevant properties of both the boundary value problems to be addressed and the second kind integral equations to be used for their solution. Most of the results are classical and can be found, for example, in [3] and [4]. The rest are straightforward generalizations to systems of ordinary differential equations of well-known facts concerning second order boundary value problems (see, for example, [5]).

2.1. Notation and Definitions

Definition 2.1 *A linear first order system of ordinary differential equations is an expression of the form*

$$\Phi'(x) + p(x) \cdot \Phi(x) = f, \tag{1}$$

with $\Phi : [a,c] \to \mathbf{R}^n$ *in* $C^1[a,c]$, $p : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ *and* $f : [a,c] \to \mathbf{R}^n$ *continuous, and* $\mathbf{L}(\mathbf{R}^{n\times n})$ *denoting the linear space of all linear operators* $\mathbf{R}^n \to \mathbf{R}^n$.

Definition 2.2 *If* $f(x) \equiv 0$, *(1) assumes the form*

$$\Phi'(x) + p(x) \cdot \Phi(x) = 0, \tag{2}$$

and is referred to as a linear homogeneous first order system of ordinary differential equations.

Definition 2.3 *A differentiable function* $\Phi : [a,c] \to \mathbf{R}^n$ *is a solution to a linear first order boundary value problem if it satisfies an equation of the form (1), subject to boundary conditions of the form*

$$A \cdot \Phi(a) + C \cdot \Phi(c) = \gamma. \tag{3}$$

with $A, C \in \mathbf{L}(\mathbf{R}^{n\times n})$, *and* $\gamma \in \mathbf{R}^n$.

Definition 2.4 *If* $\gamma \equiv 0$, *(3) becomes*

$$A \cdot \Phi(a) + C \cdot \Phi(c) = 0, \tag{4}$$

and is referred to as a set of homogeneous boundary conditions.

Definitions 2.5-2.6 are the nonlinear analogues to Definitions 2.1 and 2.3.

Definition 2.5 *A nonlinear first order system is defined as an expression*

$$\Phi'(x) = F(\Phi(x), x), \tag{5}$$

with $\Phi : [a,c] \to \mathbf{R}^n$ *in* $C^1[a,c]$, $F : \mathbf{R}^{n+1} \to \mathbf{R}^n$ *continuous.*

Definition 2.6 *A differentiable function $\Phi : [a,c] \to \mathbf{R}^n$ is a solution to a nonlinear first order boundary value problem if it satisfies an equation of the form (5), subject to boundary conditions of the form*

$$A \cdot \Phi(a) + C \cdot \Phi(c) = \gamma, \tag{6}$$

with $A, C \in \mathbf{L}(\mathbf{R}^{n\times n})$, $\gamma \in \mathbf{R}^n$.

Definition 2.7 *A continuous function $G(x,t) : [a,c]\times[a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ is the Green's function for a boundary value problem (1), (4) if*

1. $\frac{\partial G(x,t)}{\partial x}$ *is continuous except at $x = t$,*
2. $G(x+0,x) - G(x-0,x) = I_n$ *for all $x \in [a,c]$,*
3. $\frac{\partial}{\partial x}G(x,t) + p(x)\cdot G(x,t) = 0$ *for all $x,t \in [a,c], x \neq t$,*
4. $A \cdot G(a,t) + C \cdot G(c,t) = 0$ *for all $t \in [a,c]$.*

Remark 2.1 Green's functions are the principal analytical tools which enable boundary value problems to be solved as second kind integral equations. However, Green's functions are known or computable for very few problems (1), (4). Fortunately, we can use one of the known Green's functions when constructing the second kind integral equation for a particular boundary value problem. When a Green's function unrelated to a problem (1), (4) is used to convert that problem to an integral equation, we will refer to this Green's function as a background Green's function.

Definition 2.8 *A function $\Upsilon : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ is called a fundamental matrix for (2) if it is nonsingular and*

$$\Upsilon'(x) + p(x)\cdot\Upsilon(x) = 0 \tag{7}$$

for all $x \in [a,c]$.

We define boundary condition matrices D and D_N to be used in theorems in the remainder of Section II.

Definition 2.9 *Given a fundamental solution matrix Υ of the system (2), and a pair of matrices A, C given by (3), the boundary condition matrices $D, D_N \in \mathbf{L}(\mathbf{R}^{n\times n})$ are defined by the formulae*

$$D = A\cdot\Upsilon(a) + C\cdot\Upsilon(c), \tag{8}$$

$$D_N = A + C. \tag{9}$$

We define a residual mapping K and Newton iterates δ_k to be used in a Newton method for nonlinear boundary value problems.

Definition 2.10 *Given functions $G_0, p_0 : [a,c]\times[a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$, we define the residual mapping $K : \mathbf{R}^{n+1} \to \mathbf{R}^n$ by the formula*

$$K(\sigma(x),x) = \sigma(x) - p_0(x)\cdot\int_a^c G_0(x,t)\cdot\sigma(t)\,dt - F\left(\int_a^c G_0(x,t)\cdot\sigma(t)\,dt, x\right). \tag{10}$$

Definition 2.11 *For any continuous $\sigma_0 : [a,c] \to \mathbf{R}^n$, we refer to the continuous functions $\delta_k : [a,c] \to \mathbf{R}^n$ as* Newton iterates *if for each $k = 1, 2, \ldots$,*

$$\delta_k = \sigma_{k+1}(x) - \sigma_k(x), \tag{11}$$

with each continuous $\sigma_k : [a,c] \to \mathbf{R}^n$ recursively defined via the formula

$$\frac{\partial K(\sigma_k(x), x)}{\partial \sigma_k} \cdot (\sigma_{k+1}(x) - \sigma_k(x)) = \; -K(\sigma_k(x), x), \quad k = 0, 1, \cdots. \tag{12}$$

Finally, we define a transposition operator to be used in the paper.

Definition 2.12 *Given an interval $[b_1, b_2] \subset \mathbf{R}$ and a mapping $\chi : L^2[b_1, b_2] \to \mathbf{L}(\mathbf{R}^{n \times n})$, the transpose $\chi^T : (L^2[b_1, b_2])^n \to \mathbf{R}^n$ of χ is defined by the formula*

$$\chi^T(\sigma) = \int_{b_1}^{b_2} \chi(t) \cdot \sigma(t)\, dt, \tag{13}$$

with $\sigma \in (L^2)^n$.

2.2. Green's Functions for First Order Systems

Theorems 2.1-2.8 provide the tools for the conversion of first order systems of differential equations into second kind integral equations. Theorems 2.1, 2.2, 2.5 and 2.6 are well known and can be found, for example, in [3] and [4]. The authors failed to locate the remaining theorems in the literature.

Theorems 2.1-2.2 provide conditions for the existence and uniqueness of solutions to (1), (4).

Theorem 2.1 *For any continuous function $p : [a,c] \to L(\mathbf{R}^{n \times n})$, the homogeneous first order system (2) has exactly n linearly independent solutions.*

Theorem 2.2 *If the matrix D defined by (8) is nonsingular, then there is a unique solution Φ to the equation (1) satisfying homogeneous boundary conditions (4). Furthermore, the solution to the homogeneous equation (2) satisfying homogeneous boundary conditions (4) is $\Phi(x) \equiv 0$.*

The purpose of the following two theorems is to permit the conversion of problems with inhomogeneous boundary conditions to those with homogeneous ones. Theorem 2.3 concerns linear problems of the form (1), (3); Theorem 2.4 concerns nonlinear problems of the form (5), (6).

Theorem 2.3 *If the boundary condition matrices D, D_N defined by (8), (9) are both nonsingular, then the solution to the problem (1), (3) is given by the formula*

$$\Phi(x) = \tilde{\Phi}(x) + \nu, \tag{14}$$

with $\nu \in \mathbf{R}^n$ given by the formula

$$\nu = (A + C)^{-1} \cdot \gamma, \tag{15}$$

and $\tilde{\Phi} : [a, c] \to \mathbf{R}^n$ in $C^1[a, c]$ the solution to the first order system

$$\tilde{\Phi}'(x) + p(x) \cdot \tilde{\Phi}(x) = f(x) - p(x) \cdot \nu, \tag{16}$$

satisfying homogeneous boundary conditions (4).

Proof. Since the matrix D is nonsingular, it immediately follows from Theorem 2.2 that there exists a unique $\tilde{\Phi}$ satisfying the equation (16). Substituting (14) into boundary conditions (3), we obtain

$$A \cdot (\tilde{\Phi}(a) + \nu) + C \cdot (\tilde{\Phi}(c) + \nu) = \gamma. \tag{17}$$

Now, (15) is easily obtained from the combination of (17) and (4), while (16) is a result of substituting (14) into (1). □

Theorem 2.4 *If there exists a unique solution $\Phi : [a, c] \to \mathbf{R}^n$ to the problem (5), (6), and if the matrix D_N defined by (9) is nonsingular, then Φ is given by the formula*

$$\Phi(x) = \tilde{\Phi}(x) + \nu, \tag{18}$$

with $\nu \in \mathbf{R}^n$ given by

$$\nu = (A + C)^{-1} \cdot \gamma, \tag{19}$$

and $\tilde{\Phi} : [a, c] \to \mathbf{R}^n \in C^1[a, c]$ the solution to the nonlinear boundary value problem

$$\tilde{\Phi}'(x) = F(\tilde{\Phi} + \nu, x) \tag{20}$$

with homogeneous boundary conditions (4).

Proof. Substituting (18) into boundary conditions (6) we obtain

$$A \cdot (\tilde{\Phi}(a) + \nu) + C \cdot (\tilde{\Phi}(c) + \nu) = \gamma. \tag{21}$$

Now, (19) is easily obtained from the combination of (21) and (4), while (20) is a result of substituting (18) into (5). □

Theorem 2.5 provides an explicit construction for the Green's function for a boundary value problem with a known fundamental matrix Υ. Given a Green's function for a homogeneous problem (2), (4), Theorem 2.6 provides an explicit solution for the inhomogeneous problem (1), (4).

Theorem 2.5 *If the matrix D defined by (8) is nonsingular, then there exists a unique Green's function $G : [a, c] \times [a, c] \to \mathbf{L}(\mathbf{R}^{n \times n})$ for (2), (4). G is given by the formula*

$$G(x, t) = \begin{cases} \Upsilon(x) \cdot (\Upsilon^{-1}(t) + J(t)) & (t \le x), \\ \Upsilon(x) \cdot J(t) & (t \ge x), \end{cases} \tag{22}$$

with $J : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ *given by the formula*

$$J(t) = -D^{-1} \cdot C \cdot \Upsilon(c) \cdot \Upsilon^{-1}(t), \tag{23}$$

and $\Upsilon : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ *the fundamental matrix for (2) (see Definition 2.8).*

Theorem 2.6 *Given a Green's function for the problem (2), (4), the solution* Φ *for the problem (1), (4) can be obtained via the formula*

$$\Phi(x) = \int_a^c G(x,t) \cdot f(t)\, dt. \tag{24}$$

The following two theorems are the principal analytical tools used in this paper. Theorem 2.7 is used to reduce a linear boundary value problem (1), (4) to a second kind integral equation, even when the Green's function for the problem is not available; Theorem 2.8 is used in the same fashion to reduce nonlinear boundary value problems (5), (4) to nonlinear second kind integral equations.

Theorem 2.7 *Suppose* $p_0 : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ *is continuous,* $\Upsilon_0 : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ *is the fundamental matrix for the equation*

$$\Phi'(x) + p_0(x) \cdot \Phi(x) = 0, \tag{25}$$

and $G_0 : [a,c] \times [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ *is the Green's function for the boundary value problem (25), (4). Suppose further that the matrix* D *defined by (8) and the matrix* $D_0 \in \mathbf{L}(\mathbf{R}^{n\times n})$ *defined by the formula*

$$D_0 = A \cdot \Upsilon_0(a) + C \cdot \Upsilon_0(c), \tag{26}$$

are both nonsingular. Then the solution Φ *to the problem (1), (4) can be obtained via the formula*

$$\Phi(x) = \int_a^c G_0(x,t) \cdot \sigma(t)\, dt, \tag{27}$$

with $\sigma : [a,c] \to \mathbf{R}^n$ *the solution to the second kind integral equation*

$$\sigma(x) + [p(x) - p_0(x)] \cdot \int_a^c G_0(x,t) \cdot \sigma(t)\, dt = f(x). \tag{28}$$

Proof. By Theorem 2.2, if matrices D, D_0 are nonsingular then the problems (1), (4) and (25), (4) have unique solutions, and therefore the background Green's function G_0 is also unique, and is defined by Theorem 2.4. Now, (28) is obtained by substituting (27) into (1). □

Remark 2.2 If $p_0(x) = p(x)$, then the solution to equation (28) is trivially $\sigma = f$. Our working assumption is that for some background problem (25), (4), the Green's function is known or computable, but that for the original differential equation (1), (4) the Green's function is unavailable.

Theorem 2.8 *Suppose $\Phi : [a,c] \to \mathbf{R}^n$ is the unique solution to (5), (4). Suppose further that $p_0 : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ is continuous, and $\Upsilon_0 : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ is a fundamental matrix for the equation*

$$\Phi'(x) + p_0(x) \cdot \Phi(x) = 0, \tag{29}$$

and $G_0 : [a,c] \times [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ is the Green's function for the boundary value problem (25), (4). Suppose finally that the matrix D_0 defined by the formula

$$D_0 = A \cdot \Upsilon_0(a) + C \cdot \Upsilon_0(c) \tag{30}$$

is nonsingular. Then Φ can be obtained via the formula

$$\Phi(x) = \int_a^c G_0(x,t) \cdot \sigma(t)\, dt, \tag{31}$$

with $\sigma : [a,c] \to \mathbf{R}^n$ the solution to the second kind integral equation

$$\sigma(x) - p_0(x) \cdot \int_a^c G_0(x,t) \cdot \sigma(t)\, dt = F\left(\int_a^c G_0(x,t) \cdot \sigma(t)\, dt, x\right). \tag{32}$$

Proof. Since D_0 is nonsingular, the background Green's function G_0 is unique, and therefore Φ can be obtained from (31). Now, (32) is obtained by substituting (31) into (5). □

2.3. Green's Functions for Particular Equations

Lemmas 2.1 - 2.4 of this subsection provide fundamental matrices and Green's functions for two particular types of boundary value problems. Lemmas 2.1, 2.2 are easily verified by substituting formulae (34), (35) into (7), (22). Similarly, Lemmas 2.3, 2.4 are verified by substituting formulae (37), (38) into (7), (22).

Lemma 2.1 *A fundamental matrix Υ_0 for the equation*

$$\Phi'(x) = 0 \tag{33}$$

is given by the formula

$$\Upsilon_0(x) = I_n, \tag{34}$$

with n the dimensionality of the problem (33), and $x \in [a,c]$ (in accordance with standard practice, I_n denotes the unity operator $\mathbf{R}^n \to \mathbf{R}^n$).

Lemma 2.2 *The Green's function G_0 corresponding to the equation (33) subject to boundary conditions (4) is given by the formula*

$$G_0(x,t) = \begin{cases} I_n - (A+C)^{-1} \cdot C & (t \le x), \\ -(A+C)^{-1} \cdot C & (t \ge x). \end{cases} \tag{35}$$

Lemma 2.3 *For any $\lambda \in R$, a fundamental matrix Υ_0 for the equation*

$$\Phi'(x) + \lambda \cdot \Phi(x) = 0 \tag{36}$$

is given by the formula

$$\Upsilon_0(x) = e^{-\lambda x} \cdot I_n, \tag{37}$$

with n the dimensionality of the problem (36), and $x \in [0,1]$.

Lemma 2.4 *The Green's function G_0 corresponding to the equation (36) subject to boundary conditions (4) is given by the formula*

$$G_0(x,t) = \begin{cases} e^{\lambda(t-x)} \cdot I_n - e^{\lambda(t-1)} \cdot (A + e^{-\lambda} \cdot C)^{-1} \cdot C & (t \le x), \\ -e^{\lambda(t-x-1)} \cdot (A + e^{-\lambda} \cdot C)^{-1} \cdot C & (t \ge x). \end{cases} \tag{38}$$

2.4. Linear Transformations for Problems with Singular D_0 or D_N

The purpose of Theorem 2.9 is to permit the conversion of a problem (1), (3) to a second kind integral equation (28). For most problems, Theorems 2.3 and 2.7 allow such a conversion, but Theorem 2.3 cannot be used when the matrix D_N defined by (9) is singular, while Theorem 2.7 cannot be used when the matrix D_0 defined by (26) is singular. We remove these obstacles in this subsection by providing a scheme which reduces a problem of the form (1), (3) with singular matrices D_0, D_N to a problem of the same form with nonsingular D_0, D_N.

Theorem 2.10 generalizes Theorem 2.9; it permits the conversion of nonlinear problems of the form (5), (6) to nonlinear integral equations of the form (32).

Remark 2.3 If only the matrix D_0 is singular, one can always choose a new background Green's function G_0 for which D_0 will be nonsingular. However, we have found that for most problems it is easier to develop a transformation of the type described in this subsection than to develop an alternate background Green's function.

Theorem 2.9 *Suppose $\Phi : [a,c] \to \mathbf{R}^n$ is the unique solution to the problem (1), (4). Suppose further that $\Upsilon_0 : [a,c] \to L(\mathbf{R}^{n\times n})$ is a fundamental matrix for the background equation (25). Suppose finally that there exists $\Psi : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ such that $\Psi \in C^1[a,c]$, $\det \Psi(x) \neq 0$ for all $x \in [a,c]$, and the matrix*

$$\overline{D}_0 = A \cdot \Psi(a) \cdot \Upsilon_0(a) + C \cdot \Psi(c) \cdot \Upsilon_0(c) \tag{39}$$

is nonsingular. Then the equation

$$\Gamma'(x) + \Psi^{-1}(x) \cdot (\Psi'(x) + p(x) \cdot \Psi(x)) \cdot \Gamma(x) = \Psi^{-1}(x) \cdot f(x). \tag{40}$$

subject to boundary conditions

$$A \cdot \Psi(a) \cdot \Gamma(a) + C \cdot \Psi(c) \cdot \Gamma(c) = 0. \tag{41}$$

has a unique solution $\Gamma : [a,c] \to \mathbf{R}^n$, and

$$\Phi(x) = \Psi(x) \cdot \Gamma(x), \tag{42}$$

for all $x \in [a,c]$.

Proof. We immediately obtain (41) by substituting (42) into (4). Now, substituting (42) and its derivative into (1), we get

$$\Psi'(x) \cdot \Gamma(x) + \Psi(x) \cdot \Gamma'(x) + p(x) \cdot \Psi(x) \cdot \Gamma(x) = f(x), \tag{43}$$

and obtain (40) by combining (43) with the fact that $\Psi(x)$ is nonsingular for all $x \in [a, c]$. □

Remark 2.4 Clearly, the transformed problem (40), (41) satisfies the conditions of Theorem 2.7, as $\overline{D}_0$ defined by (39) is nonsingular. However, for many problems, the boundary condition matrix $\overline{D}_N$ defined by the formula

$$\overline{D}_N = A \cdot \Psi(a) + C \cdot \Psi(c). \tag{44}$$

is singular, and therefore the transformed problem fails to satisfy the conditions of Theorem 2.3. If one needs to use the results of both The rem 2.7 and 2.3, one must choose a transformation Ψ such that both $\overline{D}_0$ and $\overline{D}_N$ are nonsingular.

Of course, it is easier to choose transformations Ψ when $\overline{D}_0 = \overline{D}_N$. This is true when the background Green's function is chosen to correspond to the equation $\Phi' = 0$. By Lemma 2.1, the fundamental matrix for this equation is $\Upsilon \equiv I_n$; the equivalence for this fundamental matrix of (39) and (44) is readily apparent.

Theorem 2.10 is the nonlinear analogue of Theorem 2.9; the proofs of the two theorems are nearly identical.

Theorem 2.10 *Suppose $\Phi : [a, c] \to \mathbf{R}^n$ is the unique solution to the problem (5), (4). Suppose further that $\Upsilon_0 : [a, c] \to L(\mathbf{R}^{n \times n})$ is a fundamental matrix for the background equation (25). Suppose finally that there exists $\Psi : [a, c] \to \mathbf{L}(\mathbf{R}^{n \times n})$ such that $\Psi \in C^1[a, c]$, $\det \Psi(x) \neq 0$ for all $x \in [a, c]$, and the matrix*

$$\overline{D}_0 = A \cdot \Psi(a) \cdot \Upsilon_0(a) + C \cdot \Psi(c) \cdot \Upsilon_0(c) \tag{45}$$

is nonsingular. Then the equation

$$\Gamma'(x) + \Psi^{-1}(x)\Psi'(x) \cdot \Gamma(x) = \Psi^{-1}(x) \cdot F(\Psi(x) \cdot \Gamma(x), x) \tag{46}$$

subject to boundary conditions

$$A \cdot \Psi(a) \cdot \Gamma(a) + C \cdot \Psi(c) \cdot \Gamma(c) = 0. \tag{47}$$

has a unique solution $\Gamma : [a, c] \to \mathbf{R}^n$, and

$$\Phi(x) = \Psi(x) \cdot \Gamma(x), \tag{48}$$

for all $x \in [a, c]$.

2.5. Newton's Method for Nonlinear Boundary Value Problems

Theorems 2.4, 2.8 of Section 2.2 reduce nonlinear boundary value problems of the form (5), (6) to nonlinear second kind integral equations of the form (32). In this subsection, we describe the convergence properties of the well-known Newton's method as applied to the latter (Theorem 2.12), and reduce each step of Newton's algorithm to the solution of a *linear* boundary value problem of the form (1), (4) (Theorem 2.11).

Theorem 2.11 permits each Newton iterate δ_k defined by (11) to be expressed as the solution to a second kind integral equation.

Theorem 2.11 *Suppose $K : \mathbf{R}^{n+1} \to \mathbf{R}^n$ in $C^1[a,c]$ defined by (10) is Fréchet differentiable at every point $x \in [a,c]$, and $\Phi_k : [a,c] \to \mathbf{R}^n$ is defined for all $k = 0, 1, \ldots$ via the formula*

$$\Phi_k(x) = \int_a^c G_0(x,t) \cdot \sigma_k(t)\, dt, \tag{49}$$

with $\sigma_k : [a,c] \to \mathbf{R}^n$ defined by (12), and $G_0 : [a,c] \times [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ the Green's function for (25), (4). Then the Newton iterates $\delta_k : [a,c] \to \mathbf{R}^n \in C^0[a,c]$ given by Definition 2.14 satisfy the equation

$$\delta_k(x) + \Omega_k(x) \cdot \int_a^c G_0(x,t) \cdot \delta_k(t)\, dt = g_k(x) \tag{50}$$

for all $k = 0, 1, \ldots$, with $\Omega_k : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ defined for all $k = 0, 1, \ldots$ by the formula

$$\Omega_k(x) = -\frac{\partial F(\Phi_k(x), x)}{\partial \Phi_k} - p_0(x), \tag{51}$$

and $g_k : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ defined for all $k = 0, 1, \ldots$ by the formula

$$g_k(x) = p_0(x) \cdot \Phi_k(x) + F(\Phi_k(x), x) - \sigma_k(x). \tag{52}$$

Proof. (50) is obtained by substituting the Fréchet derivative of the function K into (12), and substituting (49), (51), (52) into the resulting equation. □

The convergence properties of Newton's method have been thoroughly studied. Theorem 2.12 is one fundamental result, and can be found, for example, in [12] (in a slightly different form).

Theorem 2.12 *Suppose Φ is the unique solution to (5), (4), $\tilde{\sigma}$ is the solution to (32), and δ is the unique solution to (50) (so that the linearization (50) to the equation (32) is nonsingular at $\tilde{\sigma}$). Then there exists $\epsilon > 0$ such that for any $\sigma_0 : [a,c] \to \mathbf{R}^n$ satisfying the condition*

$$\|\sigma_0 - \tilde{\sigma}\| \leq \epsilon \tag{53}$$

and Newton iterates $\sigma_k : [a,c] \to \mathbf{R}^n$ defined by (11),

1. *$\|\sigma_k - \tilde{\sigma}\| \leq \epsilon$ for all $k = 1, 2, \ldots$,*
2. *$\lim_{k\to\infty} \sigma_k = \tilde{\sigma}$,*
3. *σ_k converges to $\tilde{\sigma}$ quadratically.*

2.6. A Lemma from Linear Algebra

Given a perturbation of the unity operator $I_{(L^2)^n} : (L^2)^n \to (L^2)^n$, the Lemma 2.5 provides its inverse. It is normally used when the rank of the perturbation is low, is a particular case of the Sherman-Morrison formula (see, for example, [9]), and is easy to verify directly.

Lemma 2.5 *For any two vectors $U, V \in (L^2)^{n\times n}$ such that $V^T \cdot U \neq I_n$,*

$$(I_{(L^2)^n} - U \cdot V^T)^{-1} = I_{(L^2)^n} + U \cdot (I_n - V^T \cdot U)^{-1} \cdot V^T. \tag{54}$$

III. The Analytical Apparatus

In the remainder of this paper, we assume that the solution to the problem (1), (4) is being sought on the interval $[a, c]$, and that b is some intermediate point ($a < b < c$). The fundamental observation on which Algorithm A is based is that the solution to the integral equation (28) on the entire domain $[a, c]$ can easily be constructed from the solutions of two independent integral equations, one defined on $[a, b]$ and one on $[b, c]$. This leads naturally to a recursive algorithm, in which independent solutions on a large number of subintervals are successively merged until the full solution is obtained. A precise formulation of the construction and the resulting numerical scheme will require some notation.

3.1. Notation We will denote the subintervals $[a, b]$ and $[b, c]$ of $[a, c]$ by A and B, respectively. For convenience, we write the integral equation (28) in the form

$$\sigma(x) + \tilde{p}(x) \cdot \int_a^c G_0(x, t) \cdot \sigma(t)\, dt = f(x), \tag{55}$$

with $\tilde{p}(x) = p(x) - p_0(x)$, and $G_0 : [a, c] \times [a, c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ the background corresponding to the equation (25) subject to boundary conditions (4).

We define the operator $P : (L^2[a, c])^n \to (L^2[a, c])^n$ corresponding to (55) by the formula

$$P(\sigma)(x) = \sigma(x) + \tilde{p}(x) \cdot \int_a^c G_0(x, t) \cdot \sigma(t)\, dt, \tag{56}$$

so that we have

$$P\sigma = f. \tag{57}$$

We will require the four operators

$$\begin{aligned}
P_{AA} &: (L^2[a, b])^n \to (L^2[a, b])^n, \\
P_{AB} &: (L^2[b, c])^n \to (L^2[a, b])^n, \\
P_{BA} &: (L^2[a, b])^n \to (L^2[b, c])^n, \\
P_{BB} &: (L^2[b, c])^n \to (L^2[b, c])^n,
\end{aligned}$$

defined by the formulae

$$P_{AA}(\sigma)(x) = \sigma(x) + \tilde{p}(x) \cdot \int_a^b G_0(x,t) \cdot \sigma(t)\, dt, \tag{58}$$

$$P_{AB}(\sigma)(x) = \tilde{p}(x) \cdot \int_b^c G_0(x,t) \cdot \sigma(t)\, dt, \tag{59}$$

$$P_{BA}(\sigma)(x) = \tilde{p}(x) \cdot \int_a^b G_0(x,t) \cdot \sigma(t)\, dt, \tag{60}$$

$$P_{BB}(\sigma)(x) = \sigma(x) + \tilde{p}(x) \cdot \int_b^c G_0(x,t) \cdot \sigma(t)\, dt. \tag{61}$$

We define the operator $Q : (L^2[a,c])^{n\times n} \to (L^2[a,c])^{n\times n}$ by the expression

$$Q(\chi)(x) = \chi(x) + \tilde{p}(x) \cdot \int_a^c G_0(x,t) \cdot \chi(t)\, dt. \tag{62}$$

We additionally require the four operators

$$\begin{aligned} Q_{AA} &: (L^2[a,b])^{n\times n} \to (L^2[a,b])^{n\times n}, \\ Q_{AB} &: (L^2[b,c])^{n\times n} \to (L^2[a,b])^{n\times n}, \\ Q_{BA} &: (L^2[a,b])^{n\times n} \to (L^2[b,c])^{n\times n}, \\ Q_{BB} &: (L^2[b,c])^{n\times n} \to (L^2[b,c])^{n\times n}, \end{aligned}$$

defined by the formulae

$$Q_{AA}(\chi)(x) = \chi(x) + \tilde{p}(x) \cdot \int_a^b G_0(x,t) \cdot \chi(t)\, dt, \tag{63}$$

$$Q_{AB}(\chi)(x) = \tilde{p}(x) \cdot \int_b^c G_0(x,t) \cdot \chi(t)\, dt, \tag{64}$$

$$Q_{BA}(\chi)(x) = \tilde{p}(x) \cdot \int_a^b G_0(x,t) \cdot \chi(t)\, dt, \tag{65}$$

$$Q_{BB}(\chi)(x) = \chi(x) + \tilde{p}(x) \cdot \int_b^c G_0(x,t) \cdot \chi(t)\, dt. \tag{66}$$

We also require the functions $\psi, v_L, v_R : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ defined by the formulae

$$\psi(x) = \tilde{p}(x) \cdot \Upsilon_0(x), \tag{67}$$

$$v_L(t) = \Upsilon_0^{-1}(t) + J_0(t), \tag{68}$$

$$v_R(t) = J_0(t), \tag{69}$$

with Υ_0 the fundamental matrix for equation (25), and $J_0 : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ defined by the formula

$$J_0(t) = -D_0^{-1} \cdot C \cdot \Upsilon_0(c) \cdot \Upsilon_0^{-1}(t), \tag{70}$$

with the matrix D_0 defined by (26), and the matrix C given by (4).

Given a function $f \in (L^2[a,c])^n$, we will follow the convention of denoting its restriction to A and B by $f_{|A}$ and $f_{|B}$, respectively. Similarly, given a function $\psi \in (L^2[a,c])^{n\times n}$, we will denote its restriction to A and B by $\psi_{|A}$ and $\psi_{|B}$, respectively. Assuming that the operators P_{AA}, P_{BB} are nonsingular, we define the functions $\eta_A : A \to \mathbf{R}^n$, $\eta_B : B \to \mathbf{R}^n$ via the formulae

$$\eta_A = P_{AA}^{-1}(f_{|A}), \tag{71}$$

$$\eta_B = P_{BB}^{-1}(f_{|B}). \tag{72}$$

Similarly, assuming that the operators Q, Q_{AA}, Q_{BB} are nonsingular, we then define the mappings

$$\begin{aligned} \chi &: [a,c]^{n\times n} \to \mathbf{L}(\mathbf{R}^{n\times n}), \\ \phi_A &: A^{n\times n} \to \mathbf{L}(\mathbf{R}^{n\times n}), \\ \phi_B &: B^{n\times n} \to \mathbf{L}(\mathbf{R}^{n\times n}), \end{aligned}$$

via the formulae

$$\chi = Q^{-1}(\psi), \tag{73}$$

$$\phi_A = Q_{AA}^{-1}(\psi_{|A}), \tag{74}$$

$$\phi_B = Q_{BB}^{-1}(\psi_{|B}). \tag{75}$$

Finally, we will define six matrices $\alpha_L^A, \alpha_R^A, \alpha_L^B, \alpha_R^B, \alpha_L, \alpha_R \in \mathbf{L}(\mathbf{R}^{n\times n})$ by the formulae

$$\alpha_L^A = \int_a^b v_L(t) \cdot \phi_A(t)\, dt\ , \tag{76}$$

$$\alpha_R^A = \int_a^b v_R(t) \cdot \phi_A(t)\, dt\ , \tag{77}$$

$$\alpha_L^B = \int_b^c v_L(t) \cdot \phi_B(t)\, dt\ , \tag{78}$$

$$\alpha_R^B = \int_b^c v_R(t) \cdot \phi_B(t)\, dt\ , \tag{79}$$

$$\alpha_L = \int_a^c v_L(t) \cdot \chi(t)\, dt\ , \tag{80}$$

$$\alpha_R = \int_a^c v_R(t) \cdot \chi(t)\, dt\ , \tag{81}$$

and six vectors $\delta_L^A, \delta_R^A, \delta_L^B, \delta_R^B, \delta_L, \delta_R \in \mathbf{R}^n$ via the formulae

$$\delta_L^A = \int_a^b v_L(t) \cdot \eta_A(t)\, dt, \tag{82}$$

$$\delta_R^A = \int_a^b v_R(t) \cdot \eta_A(t)\, dt, \tag{83}$$

$$\delta_L^B = \int_b^c v_L(t) \cdot \eta_B(t)\, dt, \tag{84}$$

$$\delta_R^B = \int_b^c v_R(t) \cdot \eta_B(t)\, dt, \tag{85}$$

$$\delta_L = \int_a^c v_L(t) \cdot \sigma(t)\, dt, \tag{86}$$

$$\delta_R = \int_a^c v_R(t) \cdot \sigma(t)\, dt, \tag{87}$$

with σ the solution to equation (57).

3.2. Analysis of the operators P_{AB}, P_{BA}

In this subsection, we observe that each of the operators P_{AB} and P_{BA} is of rank n, and give simple expressions for these operators.

Lemma 3.1 *In the notation of the preceding subsection,*

$$P_{AB} = \psi_{|A} \cdot v_R^T, \tag{88}$$

$$P_{BA} = \psi_{|B} \cdot v_L^T. \tag{89}$$

Proof. We obtain (88) by observing that $x \leq t$ for any $x \in [a,b], t \in [b,c]$, and by using (59) and (69). Similarly, (89) follows from the combination of (60), (69) and observing that $x \geq t$ for any $x \in [b,c]$, $t \in [a,b]$. □

3.3. Recursive solution of the integral equation (57)

We now consider the original integral equation (57)

$$P\sigma = f.$$

The main result of this subsection is the following lemma, which constructs the solution σ of equation (57) from η_A, η_B of equations (71) and (72).

Lemma 3.2 *If, in the notation of Section 3.1, all six operators P, P_{AA}, P_{BB}.Q, Q_{AA}, Q_{BB} are nonsingular, and the matrices $\Delta_1, \Delta_2 \in \mathbf{L}(\mathbf{R}^{n\times n})$ defined by the formulae*

$$\Delta_1 = I_n - \alpha_L^A \cdot \alpha_R^B, \tag{90}$$

$$\Delta_2 = I_n - \alpha_R^B \cdot \alpha_L^A, \tag{91}$$

are also nonsingular, then

$$\sigma_{|A} = \eta_A + \phi_A \cdot \Delta_2^{-1} \cdot (\alpha_R^B \cdot \delta_L^A - \delta_R^B), \tag{92}$$

$$\sigma_{|B} = \eta_B + \phi_B \cdot \Delta_1^{-1} \cdot (\alpha_L^A \cdot \delta_R^B - \delta_L^A). \tag{93}$$

Proof. Using definitions (56) - (61), the integral equation

$$P\sigma = f$$

can be rewritten in the form

$$P_{AA}(\sigma_{|A}) + P_{AB}(\sigma_{|B}) = f_{|A}, \tag{94}$$

$$P_{BA}(\sigma_{|A}) + P_{BB}(\sigma_{|B}) = f_{|B}. \tag{95}$$

The expansions (88) and (89) for P_{AB} and P_{BA}, respectively, can then be used to obtain an explicit solution to the coupled equations (94) and (95) in terms of the functions η_A, η_B, ϕ_A, and ϕ_B defined by (71), (72), (74), (75), respectively. Indeed, applying the operator P_{AA}^{-1} to equation (94) and the operator P_{BB}^{-1} to equation (95), we have

$$\sigma_{|A} + P_{AA}^{-1} \cdot P_{AB}(\sigma_{|B}) = P_{AA}^{-1}(f_{|A}), \tag{96}$$

$$P_{BB}^{-1} \cdot P_{BA}(\sigma_{|A}) + \sigma_{|B} = P_{BB}^{-1}(f_{|B}). \tag{97}$$

Substituting (88) and (89) into (96) and (97) yields the formulae

$$\sigma_{|A} + P_{AA}^{-1} \cdot \psi_{|A} \cdot v_R^T \cdot \sigma_{|B} = \eta_A, \tag{98}$$

$$P_{BB}^{-1} \cdot \psi_{|B} \cdot v_L^T \cdot \sigma_{|A} + \sigma_{|B} = \eta_B, \tag{99}$$

or

$$\sigma_{|A} + \phi_A \cdot v_R^T \cdot \sigma_{|B} = \eta_A, \tag{100}$$

$$\phi_B \cdot v_L^T \cdot \sigma_{|A} + \sigma_{|B} = \eta_B, \tag{101}$$

where we have used the definitions (74), (75) for ϕ_A and ϕ_B, respectively. Now, multiplying (101) by $\phi_A \cdot v_R^T$ and subtracting it from (100), we obtain

$$(I_{(L^2)^n} - \phi_A \cdot v_R^T \cdot \phi_B \cdot v_L^T) \cdot \sigma_{|A} = \eta_A - \phi_A \cdot v_R^T \cdot \eta_B. \tag{102}$$

Similarly, multiplying (100) by $\phi_B \cdot v_L^T$ and subtracting it from (101) results in the equation

$$(I_{(L^2)^n} - \phi_B \cdot v_L^T \cdot \phi_A \cdot v_R^T) \cdot \sigma_{|B} = \eta_B - \phi_B \cdot v_L^T \cdot \eta_A. \tag{103}$$

Due to (76), (79), and (82), (85) we can rewrite these equations in the form

$$(I_{(L^2)^n} - \phi_A \cdot (\alpha_R^B \cdot v_L^T)) \cdot \sigma_{|A} = \eta_A - \phi_A \cdot \delta_R^B, \tag{104}$$

$$(I_{(L^2)^n} - \phi_B \cdot (\alpha_L^A \cdot v_R^T)) \cdot \sigma_{|B} = \eta_B - \phi_B \cdot \delta_L^A. \tag{105}$$

By application of Lemma 2.5, we obtain

$$\sigma_{|A} = (I_{(L^2)^n} + \phi_A \cdot (I_n - \alpha_R^B \cdot v_L^T \cdot \phi_A)^{-1} \cdot \alpha_R^B \cdot v_L^T) \cdot (\eta_A - \phi_A \cdot \delta_R^B), \tag{106}$$

$$\sigma_{|B} = (I_{(L^2)^n} + \phi_B \cdot (I_n - \alpha_L^A \cdot v_R^T \cdot \phi_B)^{-1} \cdot \alpha_L^A \cdot v_R^T) \cdot (\eta_B - \phi_B \cdot \delta_L^A). \tag{107}$$

The equations (92), (93) are now obtained from equations (106), (107) and equations (90). (91). □

Remark 3.1 Suppose that b_1 and b_2 are a pair of real numbers such that $a < b_1 < b_2 < c$, and that the interval $[b_1, b_2]$ is denoted by C. We will denote by P_{CC} the restriction of the operator P to the interval C, and denote by Q_{CC} the restriction of the operator Q to the interval C. Assuming that P_{CC}, Q_{CC} are nonsingular, we define the functions $\eta_C : C \to \mathbf{R}^n, \phi_C : C \to \mathbf{L}(\mathbf{R}^{n \times n})$ by

$$\eta_C = P_{CC}^{-1}(f_{|C}), \tag{108}$$

$$\phi_C = Q_{CC}^{-1}(\psi_{|C}) \, . \tag{109}$$

By applying the above lemma twice (once for the subinterval $[a, b_1]$ and once for $[a, b_2]$), we may easily observe that there exists $\lambda \in \mathbf{R}^n$ such that

$$\sigma(x) = \eta_C(x) + \phi_C(x) \cdot \lambda \tag{110}$$

for all $x \in C$. The exact expression for the vector λ is complicated, but irrelevant for the purposes of this paper. The existence of a relation of the form (110), however, will be critically important in Section 4.

3.4. Further Analytical Results

We now collect a number of identities which are necessary for Algorithm A, to be presented in Section 4. Corollary 3.1 is similar to Lemma 3.2, but uses the matrix valued function ψ in place of the vector valued function f to obtain an analytical expression for the function χ defined by (73).

Corollary 3.1 *If, in the notation of Section 3.1, all six operators $P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB}$ are nonsingular, then*

$$\chi_{|A} = \phi_A \cdot \Delta_2^{-1} \cdot (I_n - \alpha_R^B), \tag{111}$$

$$\chi_{|B} = \phi_B \cdot \Delta_1^{-1} \cdot (I_n - \alpha_L^A). \tag{112}$$

with the matrices α_L^A and α_R^B defined by equations (76) and (79), and the matrices Δ_1, Δ_2 defined by equations (90) and (91).

Proof. Substituting in equations (106), (107) the functions ϕ_A, ϕ_B defined by (74), (75) for the functions η_A, η_B defined by (71), (72), and the matrices α_L^A, α_R^B defined by (76), (79) for the vectors δ_L^A, δ_R^B defined by (82), (85), we obtain

$$\chi_{|A} = \phi_A - \phi_A \cdot \alpha_R^B + \phi_A \cdot \Delta_2^{-1} \cdot \alpha_R^B \cdot \alpha_L^A - \phi_A \cdot \Delta_2^{-1} \cdot \alpha_R^B \cdot \alpha_L^A \cdot \alpha_R^B, \tag{113}$$

$$\chi_{|B} = \phi_B - \phi_B \cdot \alpha_L^A + \phi_B \cdot \Delta_1^{-1} \cdot \alpha_L^A \cdot \alpha_R^B - \phi_B \cdot \Delta_1^{-1} \cdot \alpha_L^A \cdot \alpha_R^B \cdot \alpha_L^A. \tag{114}$$

The expressions (111), (112) are now easily obtained from the equations (113), (114). □

We will also require analytical expressions for the inner products δ_L and δ_R defined by (86), (87) in terms of the restricted inner products $\delta_l^A, \delta_l^B, \delta_r^A$ and δ_r^B defined by (82)-(85).

Corollary 3.2 *If, in the notation of Section 3.1, all six operators* $P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB}$ *are nonsingular, then*

$$\begin{aligned}\delta_L &= \int_a^c v_L(t)\cdot\sigma(t)\,dt = \int_a^b v_L(t)\cdot\sigma_{|A}(t)\,dt + \int_b^c v_L(t)\cdot\sigma_{|B}(t)\,dt \\ &= \delta_L^A + \delta_L^B + \alpha_L^A\cdot\Delta_2^{-1}\cdot(\alpha_R^B\cdot\delta_L^A - \delta_R^B) + \alpha_L^B\cdot\Delta_1^{-1}\cdot(\alpha_L^A\cdot\delta_R^B - \delta_L^A), \qquad (115)\end{aligned}$$

$$\begin{aligned}\delta_R &= \int_a^c v_R(t)\cdot\sigma(t)\,dt = \int_a^b v_R(t)\cdot\sigma_{|A}(t)\,dt + \int_b^c v_R(t)\cdot\sigma_{|B}(t)\,dt \\ &= \delta_R^A + \delta_R^B + \alpha_R^A\cdot\Delta_2^{-1}\cdot(\alpha_R^B\cdot\delta_L^A - \delta_R^B) + \alpha_R^B\cdot\Delta_1^{-1}\cdot(\alpha_L^A\cdot\delta_R^B - \delta_L^A). \qquad (116)\end{aligned}$$

Proof. Multiplying equation (92) by v_L^T and v_R^T, and equation (93) by v_L^T and v_R^T, we obtain

$$\int_a^b v_L(t)\cdot\sigma_{|A}(t)\,dt = \delta_L^A + \alpha_L^A\cdot\Delta_2^{-1}\cdot(\alpha_R^B\cdot\delta_L^A - \delta_R^B), \qquad (117)$$

$$\int_b^c v_L(t)\cdot\sigma_{|B}(t)\,dt = \delta_L^B + \alpha_L^B\cdot\Delta_1^{-1}\cdot(\alpha_L^A\cdot\delta_R^B - \delta_L^A), \qquad (118)$$

$$\int_a^b v_R(t)\cdot\sigma_{|A}(t)\,dt = \delta_R^A + \alpha_R^A\cdot\Delta_2^{-1}\cdot(\alpha_R^B\cdot\delta_L^A - \delta_R^B), \qquad (119)$$

$$\int_b^c v_R(t)\cdot\sigma_{|B}(t)\,dt = \delta_R^B + \alpha_R^B\cdot\Delta_1^{-1}\cdot(\alpha_L^A\cdot\delta_R^B - \delta_L^A). \qquad (120)$$

Now, expressions (115), (116) are easily obtained from (117)-(120). □

Corollary 3.3 is similar to Corollary 3.2, but uses χ, the matrix valued function defined by (73), in place of σ, the vector valued function defined by (57). While the two corollaries concern different objects (the vectors δ_L, δ_R in Corollary 3.2, the matrices α_L, α_R in Corollary 3.3), their proofs are nearly identical.

Corollary 3.3 *If, in the notation of Section 3.1, all six operators* $P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB}$ *are nonsingular, then*

$$\begin{aligned}\alpha_L &= \int_a^c v_L(t)\cdot\chi(t)\,dt = \int_a^b v_L(t)\cdot\chi_{|A}(t)\,dt + \int_b^c v_L(t)\cdot\chi_{|B}(t)\,dt \\ &= \alpha_L^A\cdot\Delta_2^{-1}\cdot(I_n - \alpha_R^B) + \alpha_L^B\cdot\Delta_1^{-1}\cdot(I_n - \alpha_L^A), \qquad (121)\end{aligned}$$

$$\begin{aligned}\alpha_R &= \int_a^c v_R(t)\cdot\chi(t)\,dt = \int_a^b v_R(t)\cdot\chi_{|A}(t)\,dt + \int_b^c v_R(t)\cdot\chi_{|B}(t)\,dt \\ &= \alpha_R^A\cdot\Delta_2^{-1}\cdot(I_n - \alpha_R^B) + \alpha_R^B\cdot\Delta_1^{-1}\cdot(I_n - \alpha_L^A). \qquad (122)\end{aligned}$$

Finally, combining Lemma 3.2 with the expressions (111)-(112), we have

Corollary 3.4 *Suppose that in the notation of Section 3.1, all six operators P, P_{AA}, P_{BB}, Q,Q_{AA},Q_{BB} are nonsingular. Suppose further that the function F is defined by the formula*

$$F(x) = \chi \cdot \lambda + \sigma. \tag{123}$$

with $\lambda \in \mathbf{R}^n$. Then on the interval $[a,b]$,

$$F(x) = \phi_A(x) \cdot \mu + \eta_A(x), \tag{124}$$

with $\mu \in \mathbf{R}^n$ defined by the formula

$$\mu = \Delta_2^{-1}(\lambda - \alpha_R^B \cdot (\lambda - \delta_L^A) - \delta_R^B). \tag{125}$$

Similarly, on the interval $[b,c]$,

$$F(x) = \phi_B(x) \cdot \nu + \eta_B(x), \tag{126}$$

with $\nu \in \mathbf{R}^n$ defined via the formula

$$\nu = \Delta_1^{-1} \cdot (\lambda - \alpha_L^A \cdot (\lambda - \delta_R^B) - \delta_L^A). \tag{127}$$

Proof. Restricting (123) on the subintervals A, B of $[a,c]$, respectively, we have

$$F_{|A} = \chi_{|A} \cdot \lambda + \sigma_{|A}, \tag{128}$$
$$F_{|B} = \chi_{|B} \cdot \lambda + \sigma_{|B}. \tag{129}$$

Combining (128), (129) with (92), (93), (111), (112), we obtain

$$F_{|A} = \phi_A \cdot \Delta_2^{-1} \cdot (I_n - \alpha_R^B) \cdot \lambda + (\eta_A + \phi_A \cdot \Delta_2^{-1} \cdot (\alpha_R^B \cdot \delta_L^A - \delta_R^B)), \tag{130}$$
$$F_{|B} = \phi_B \cdot \Delta_1^{-1} \cdot (I_n - \alpha_L^A) \cdot \lambda + (\eta_B + \phi_B \cdot \Delta_1^{-1} \cdot (\alpha_L^A \cdot \delta_R^B - \delta_L^A)). \tag{131}$$

Now, the expressions (125), (127) immediately follow from the comparisons of (124), (126) with (130), (131), respectively. □

IV. Description of the Algorithms

We turn now to the construction of the fast algorithm for the solution of the integral equation (57)

$$P\sigma = f\ ,$$

based on the apparatus developed in Section III. The main tool at our disposal is the ability to merge the solutions of restricted versions of the integral equation in adjacent subintervals (Lemma 3.2). As this suggests a recursive procedure, we begin by subdividing the whole interval $[a,c]$, on which the solution to (57) is sought, into a large number of subintervals. For the sake of simplicity, we assume that m is a positive integer and that $M = 2^m$ is the number

of subintervals created. The boundary points of the subintervals are then defined by a strictly increasing sequence of numbers

$$b_1, b_2, \ldots, b_M, b_{M+1}, \tag{132}$$

with $b_1 = a$ and $b_{M+1} = c$. For each $i = 1, \ldots, M$, we define the interval B_i^m via the expression

$$B_i^m = [b_i, b_{i+1}], \tag{133}$$

and create a hierarchy of intervals B_i^j by recursively merging adjacent pairs. That is, for each $j = m-1, \ldots, 1, 0$, and $i = 1, \ldots, M$, we define

$$B_i^l = B_{2i-1}^{l+1} \bigcup B_{2i}^{l+1}. \tag{134}$$

We will refer to each fixed l as a *level.* We will also refer to the two intervals B_{2i-1}^{l+1} and B_{2i}^{l+1} as *children* and to the larger interval B_i^l as a *parent.*

It is obvious that

$$B_i^l = [b_{1+(i-1)\cdot 2^{m-l}}, b_{1+i\cdot 2^{m-l}}], \tag{135}$$

and that for each level l,

$$[a, c] = \bigcup_{i=1}^{2^l} B_i^l \ . \tag{136}$$

4.1. Notation

Generalizing the notation of Section III, we will denote by $P_{i,l}$ the restriction to the interval B_i^l of the integral operator P, so that

$$P_{i,l}(\sigma)(x) \ = \ \sigma(x) + \tilde{p}(x) \cdot \int_{b_{1+(i-1)\cdot 2^{m-l}}}^{b_{1+i\cdot 2^{m-l}}} G_0(x,t) \cdot \sigma(t)\, dt \tag{137}$$

for any $\sigma \in L^2(B_i^l)^n$. Similarly, we will denote by $Q_{i,l}$ the restriction to the interval B_i^l of the integral operator Q, so that

$$Q_{i,l}(\chi)(x) \ = \ \chi(x) + \tilde{p}(x) \cdot \int_{b_{1+(i-1)\cdot 2^{m-l}}}^{b_{1+i\cdot 2^{m-l}}} G_0(x,t) \cdot \chi(t)\, dt \tag{138}$$

for any $\chi \in L^2(B_i^l)^{n\times n}$. For each B_i^l we will define the functions $\eta_{i,l} : B_i^l \rightarrow \mathbf{R}^n, \varphi_{i,l} : B_i^l \rightarrow \mathbf{L}(\mathbf{R}^{n\times n})$ as the solutions of the equations

$$P_{i,l}(\eta_{i,l}) = f_{|B_i^l}, \tag{139}$$

$$Q_{i,l}(\phi_{i,l}) = \psi_{|B_i^l}, \tag{140}$$

provided the operators $P_{i,l}, Q_{i,l}$ are nonsingular.

Remark 4.1 Suppose now that the operators $P_{i,l}, Q_{i,l}$ are nonsingular on the interval B_i^l. Then, due to (110), there exists $\lambda^{i,l} \in \mathbf{R}^n$ such that

$$\sigma(x) = \eta_{i,l}(x) + \phi_{i,l}(x) \cdot \lambda^{i,l} \tag{141}$$

for all $x \in B_i^l$.

For each $l = 0, 1, \ldots, m$, and $i = 1, 2, \ldots, 2^l$, we define the matrices $\alpha_L^{i,l}, \alpha_R^{i,l} \in \mathbf{L}(\mathbf{R}^{n \times n})$ by the formulae

$$\alpha_L^{i,l} = \int_{b_{1+(i-1)\cdot 2^{m-l}}}^{b_{1+i\cdot 2^{m-l}}} v_{L|B_i^l}(t) \cdot \phi_{i,l}(t)\, dt, \tag{142}$$

$$\alpha_R^{i,l} = \int_{b_{1+(i-1)\cdot 2^{m-l}}}^{b_{1+i\cdot 2^{m-l}}} v_{R|B_i^l}(t) \cdot \phi_{i,l}(t)\, dt, \tag{143}$$

and the vectors $\delta_L^{i,l}, \delta_R^{i,l} \in \mathbf{R}^n$ by the formulae

$$\delta_L^{i,l} = \int_{b_{1+(i-1)\cdot 2^{m-l}}}^{b_{1+i\cdot 2^{m-l}}} v_{L|B_i^l}(t) \cdot \eta_{i,l}(t)\, dt, \tag{144}$$

$$\delta_R^{i,l} = \int_{b_{1+(i-1)\cdot 2^{m-l}}}^{b_{1+i\cdot 2^{m-l}}} v_{R|B_i^l}(t) \cdot \eta_{i,l}(t)\, dt. \tag{145}$$

4.2. Discretization of the Restricted Integral Equations

Choosing an integer $p \geq 1$, we construct the p scaled Chebyshev nodes

$$\tau_i^j = \left(\frac{b_{i+1} - b_i}{2}\right) \cos\left[\frac{(2j-1)\pi}{2p}\right] + \left(\frac{b_{i+1} + b_i}{2}\right) \qquad j = 1, 2, \ldots, p \tag{146}$$

on each of the intervals B_i^m, $i = 1, 2, \ldots, M$. We then discretize the two integral equations (139), (140) via a Nyström algorithm based on p–point Chebyshev quadrature (see, for example, [11]). The resulting approximations to the functions $\eta_{i,l}$, $\phi_{i,l}$ at the nodes τ_i^j will be denoted by

$$\tilde{\eta}_{i,l} = (\tilde{\eta}_{i,l}^1, \tilde{\eta}_{i,l}^2, \ldots, \tilde{\eta}_{i,l}^p)\,,$$
$$\tilde{\phi}_{i,l} = (\tilde{\phi}_{i,l}^1, \tilde{\phi}_{i,l}^2, \ldots, \tilde{\phi}_{i,l}^p)\,,$$

respectively.

Remark 4.2 It is well-known that the order of convergence of the approximations $\tilde{\eta}_{i,l}, \tilde{\phi}_{i,l}$ to the functions $\eta_{i,l}, \phi_{i,l}$ is p. Since all subsequent steps in the construction of an approximate solution $\tilde{\sigma}$ to the integral equation (57) are analytic, the convergence rate of the full algorithm depends entirely on the parameter p. For example, by using 16 scaled Chebyshev points on each subinterval at the finest level, one obtains a sixteenth order method.

Remark 4.3 The algorithm of this section makes extensive use of the apparatus of Chebyshev interpolation, quadratures, composite quadratures, etc. This apparatus is quite well-developed, and can be found in various forms in [6], [8], [10]. For a detailed description in the form most convenient for our purposes, we refer the reader to [11].

4.3. Informal Description of the Algorithm for Linear ODEs

We begin by directly solving the two integral equations (139), (140) on each subinterval B_i^m at the finest level, as discussed in the preceding subsection. Equation (141) then shows that σ restricted to B_i^m can be expressed as a linear combination of the two solutions $\eta_{i,m}$, $\phi_{i,m}$. Thus, it remains only to determine the coefficients $\lambda^{i,m} \in \mathbf{R}^n$ for each of the M subintervals B_i^m. Fortunately, this can be done recursively. To see this, suppose that, at some coarse level $l \leq m-1$, we are given the coefficient $\lambda^{i,l}$ for the subinterval B_i^l. Then Corollary 3.4 provides formulae for the calculation of the corresponding coefficients $\lambda^{2i-1,l+1}, \lambda^{2i,l+1} \in \mathbf{R}^n$ for the two child intervals B_{2i-1}^{l+1} and B_{2i}^{l+1}, respectively. On the coarsest level, we observe that $\lambda^{0,1} = 0$. i.e. the solution of equation (139) on the whole interval $[a,c]$ is simply σ.

However, the formulae (125) and (127) of Corollary 3.4 contain the matrices $\alpha_L^{2i-1,l+1}$, $\alpha_R^{2i-1,l+1}$, $\alpha_L^{2i,l+1}$, $\alpha_R^{2i,l+1}$ and the vectors $\delta_L^{2i-1,l+1}$, $\delta_R^{2i-1,l+1}$, $\delta_L^{2i,l+1}$, $\delta_R^{2i,l+1}$. These quantities are also computed recursively but in the opposite direction, namely, from the finest level to the coarsest. They are certainly available at level m directly from the definitions (142)-(145). For the interval B_i^l at any coarser level $l \leq m-1$, Corollaries 3.2 and 3.3 describe how matrices $\alpha_L^{i,l}, \alpha_R^{i,l}$ and vectors $\delta_L^{i,l}, \delta_R^{i,l}$ are obtained from the matrices α_L, α_R and vectors δ_L, δ_R of the two child intervals.

To summarize, the algorithm consists of three parts. First, a sufficiently fine subdivision $b_1, b_2, \ldots, b_{M+1}$ of the interval $[a,c]$ is chosen so that, on each of the intervals $B_{i,m}$, the functions $\eta_{i,m}, \phi_{i,m}$ can be accurately represented by a low order Chebyshev expansion. On each of the intervals $B_{i,m}$, the equations (139) - (140) are solved (approximately) by direct inversion of the linear system arising from a Nyström discretization. Second, the matrices $\alpha_L^{i,l}, \alpha_R^{i,l}$ and vectors $\delta_L^{i,l}, \delta_R^{i,l}$ are computed in an upward sweep, beginning at the finest level m. Finally, the coefficients $\lambda^{i,l}$ are computed in a downward sweep, beginning at the coarsest level. The desired function σ is then recovered on each subinterval from equation (141).

The following is a more detailed description of the numerical procedure.

Algorithm A

Comment [Define the computational grid.]

Create $M = 2^m$ subintervals on $[a,c]$ by choosing a sequence of boundary points $b_1, b_2, \ldots, b_M, b_{M+1}$ with $b_1 = a$ and $b_{M+1} = c$. Choose the number p of Chebyshev nodes on each interval $B_i^m = [b_i, b_{i+1}]$ for $i = 1, \ldots, M$. Determine the locations of the scaled Chebyshev nodes $\tau_i^1, \tau_i^2, \ldots, \tau_i^p$ on each interval B_i^m, and evaluate the functions f, ψ at these nodes, obtaining $f_{i,m}, \psi_{i,m}$.

Step 1.

Comment [Construct the approximate solutions $\tilde{\eta}_{i,m}, \tilde{\phi}_{i,m}$ on each interval B_i^m.]

do i $= 1, 2, \ldots, M$

(1) Construct the two $p \cdot n \times p \cdot n$ linear systems on B_i^l obtained through a Nyström discretization of the corresponding integral equation.

(2) Solve the two $p \cdot n \times p \cdot n$ linear systems on B_i^l by Gaussian elimination, obtaining the values $\tilde{\eta}_{i,m}, \tilde{\phi}_{i,m}$.

end do

Step 2.

Comment [Construct the matrices $\alpha_L^{i,m}, \alpha_R^{i,m}$ and vectors $\delta_L^{i,m}, \delta_R^{i,m}$ on each interval B_i^m at the finest level.]

do i $= 1, 2, \ldots, M$

Evaluate the matrices $\alpha_L^{i,m}, \alpha_R^{i,m}$ and vectors $\delta_L^{i,m}, \delta_R^{i,m}$ using the $p-$point Chebyshev quadrature formula.

end do

Step 3 (Upward Sweep).

Comment [Construct the matrices $\alpha_L^{i,l}, \alpha_R^{i,l}$ and vectors $\delta_L^{i,l}, \delta_R^{i,l}$ for all intervals at all coarser levels $l = m-1, m-2, \ldots, 0$.]

do l= m-1, 0, -1

do i=1, 2^l

Compute the matrices $\alpha_L^{i,l}, \alpha_R^{i,l}$ and vectors $\delta_L^{i,l}, \delta_R^{i,l}$ from the corresponding data in the two child intervals $(\alpha_L^{2i-1,l+1}, \alpha_R^{2i-1,l+1}, \alpha_L^{2i,l+1}, \alpha_R^{2i,l+1}, \delta_L^{2i-1,l+1}, \delta_R^{2i-1,l+1}, \delta_L^{2i,l+1}, \delta_R^{2i,l+1})$, using the results of Corollaries 3.2 and 3.3.

end do

end do

Step 4 (Downward Sweep).

Comment [Construct the coefficient $\lambda^{i,m}$ for all intervals at the finest level.]

Set $\lambda^{0,1} = 0$.

do l=0,m-1

do i=1, 2^l

Use Corollary 3.4 to compute the coefficients $\lambda^{l+1,2i-1}, \lambda^{2i,l+1}$, for the child intervals B_{2i}^{l+1} and B_{2i-1}^{l+1} from the coefficient $\lambda^{i,l}$ of the parent interval B_i^l.

end do

end do

Step 5.

Comment [Compute the solution σ of equation (57) at the nodes $\tau_i^1, \tau_i^2, \ldots, \tau_i^p$ for each interval B_i^m at the finest level.]

do i=1, M
  do j=1,p
    Determine the values of the solution σ of equation (57) at the node τ_i^j via formula (141).
  end do
end do

Step 6.

Comment [Compute the solution ϕ of equation (1) from the values of σ.]

Evaluate the integral (27), by using composite Chebyshev quadrature
(see Remark 4.5 below).

Remark 4.4 Inspection of the above algorithm shows that the amount of work required is of the order $O(M \cdot p^3 \cdot n^3)$. Step 1 involves solving two $(p \times n) \times (p \times n)$ linear systems for each of the M intervals. Steps 2 - 5 require no more than $O(M \cdot p \cdot n^2 \cdot (\log p + n))$ operations. Since $N = M \cdot p$ is the total number of nodes in the discretization of the interval $[a, c]$, we can write the CPU time estimate in the form $O(N \cdot p^2 \cdot n^3)$. The cost of evaluating the solution Φ of the differential equation (55) from the integral representation (27) is $O(N \cdot \log p \cdot n)$ (see Remark 4.5 below).

Remark 4.5 The final step in the algorithm involves the evaluation of an integral of the form (27) at each of the Chebyshev nodes τ_i^j on each subinterval B_i^m, namely

$$\Phi(\tau_i^j) = \int_a^c G_0(\tau_i^j, t) \cdot \sigma(t)\, dt. \tag{147}$$

If these integrals were calculated independently for each τ_i^j, the amount of work required would be of the form $O(N^2 \cdot n)$, and would dominate the construction of the function Φ. In fact, this is unnecessary, for we may write

$$\begin{aligned} \Phi(\tau_i^j) \;=\; & \Upsilon(\tau_i^j) \cdot \left[\int_a^{b_i} v_L(t) \cdot \sigma(t)\, dt + \int_{b_i}^{\tau_i^j} v_L(t) \cdot \sigma(t)\, dt \right. \\ & \left. + \int_{\tau_i^j}^{b_{i+1}} v_R(t) \cdot \sigma(t)\, dt + \int_{b_{i+1}}^{c} v_R(t) \cdot \sigma(t)\, dt \right], \end{aligned} \tag{148}$$

where we have used the representation (22) and the fact that τ_i^j lies in the interval $B_i^m = [b_i, b_{i+1}]$. Step 6 can then be written in detail as follows:

Step 6 (a).

Comment [Precompute the integrals of $v_L \cdot \sigma$ and $v_R \cdot \sigma$ on each subinterval B_i^m by Chebyshev quadrature. These integrals will be denoted I_L and I_R, respectively.]

do i=1, M

$I_L(B_i^m) = \int_{b_i}^{b_{i+1}} v_L(t) \cdot \sigma(t)\, dt.$

$I_R(B_i^m) = \int_{b_i}^{b_{i+1}} v_R(t) \cdot \sigma(t)\, dt.$

end do

Step 6 (b).

Comment [March across interval from a to c, computing Φ at each node in the discretization. The variables $\mathcal{J}_L$ and $\mathcal{J}_R$ will be used to accumulate the integrals $\int_a^{b_i} v_L(t) \cdot \sigma(t)\, dt$ and $\int_{b_{i+1}}^{c} v_R(t) \cdot \sigma(t)\, dt$, respectively.

Set $\mathcal{J}_R = \sum_{i=2}^{M} I_R(B_i^m)$.
Set $\mathcal{J}_L = 0$.

do i=1, M
  do j=1,p
    For each τ_i^j, compute
    $\Phi(\tau_i^j) = \Upsilon(\tau_i^j) \cdot \left[\mathcal{J}_L + \int_{b_i}^{\tau_i^j} v_L(t) \cdot \sigma(t)\, dt + \int_{\tau_i^j}^{b_{i+1}} v_R(t) \cdot \sigma(t)\, dt + \mathcal{J}_R\right]$
  end do
  $\mathcal{J}_L = \mathcal{J}_L + I_L(B_i^m)$
  $\mathcal{J}_R = \mathcal{J}_R - I_R(B_{i+1}^m)$
end do

Thus, the amount of work required in Step 6(a) is $O(N \cdot n)$. The integrals required on each subinterval in Step 6 (b) can be computed by spectral integration (see, for example, [11]) using $O(p \cdot \log p \cdot n)$ work. The total cost is therefore of the order $O(M \cdot p \cdot \log p \cdot n)$ or $O(N \cdot \log p \cdot n)$.

4.4. Informal Description of the Algorithm for Nonlinear ODEs

The nonlinear algorithm is a straightforward application of the linear algorithm described in the previous subsection. The solution is obtained using Newton's method for nonlinear ODEs: each Newton iterate is obtained by solving the linearized problem (50) via the algorithm of the preceding subsection.

As with Algorithm A, we subdivide the interval $[a, c]$ into a large number of subintervals M: for simplicity we assume $M = 2^m$, with m a positive integer. As before the boundary points $b_1, b_2, \ldots, b_M, b_{M+1}$ are defined by (132), and the intervals $B_i^l, (1 \le l \le m), (1 \le i \le 2^l)$ by (133).

On the k^{th} step of the Newton process, Algorithm A is applied to the integral equation

$$P^k \delta_k = g_k, \tag{149}$$

with the operator $P^k : (L^2[a,c])^n \to (L^2[a,c])^n$ defined by the formula

$$P^k(\delta_k)(x) = \delta_k(x) + \Omega_k(x) \cdot \int_a^c G_0(x,t) \cdot \delta_k(t)\, dt, \tag{150}$$

with δ_k the solution of the integral equation (50), Ω_k given by (51), and g_k given by (52). The integral equations (139), (140) now assume the form

$$P_{i,l}^k(\eta_{i,l}) = g_{k|B_i^l}, \tag{151}$$

$$Q_{i,l}^k(\phi_{i,l}) = \Omega_{k|B_i^l}, \tag{152}$$

with the operator $P_{i,l}^k : (L^2[a,c])^n \to (L^2[a,c])^n$ defined by the formula

$$P_{i,l}^k(\delta_k)(x) = \delta_k(x) + \Omega_k(x) \cdot \int_{b_{1+(i-1)\cdot 2^{m-l}}}^{b_{1+i\cdot 2^{m-l}}} G_0(x,t) \cdot \delta_k(t)\, dt, \tag{153}$$

and the operator $Q_{i,l}^k : (L^2[a,c])^{n\times n} \to (L^2[a,c])^{n\times n}$ defined via the formula

$$Q_{i,l}^k(\chi)(x) = \chi(x) + \Omega_k(x) \cdot \int_{b_{1+(i-1)\cdot 2^{m-l}}}^{b_{1+i\cdot 2^{m-l}}} G_0(x,t) \cdot \chi(t)\, dt. \tag{154}$$

Once Algorithm A has computed the solution δ_k to (149), we obtain σ_{k+1} via (11), and Φ_{k+1} via (49).

The nonlinear algorithm requires an initial approximation Φ_0, Φ_0' to the solution Φ and its derivative Φ' of equation (5), and we assume that both are supplied by the calling program. σ_0 is obtained from Φ_0, Φ_0' via the identity

$$\sigma_0(x) = \Phi_0'(x) + p_0(x) \cdot \Phi_0(x). \tag{155}$$

The procedure is terminated when the stopping criterion

$$\frac{\|\delta_k\|_2}{\|\sigma_k\|_2} \le \epsilon \tag{156}$$

is satisfied, with ϵ provided by the calling program. Since Newton's method frequently fails to converge, the calling program also permits a certain maximum number of iterations, after which the algorithm stops, signalling failure.

The following is a more detailed description of the numerical procedure.

Algorithm B

Comment [Define the computational grid.]

Create $M = 2^m$ subintervals on $[a,c]$ by choosing a sequence of boundary points $b_1, b_2, \ldots, b_M, b_{M+1}$ with $b_1 = a$ and $b_{M+1} = c$. Choose the number p of Chebyshev nodes on each interval $B_i^m = [b_i, b_{i+1}]$ for $i = 1, \ldots, M$. Determine the locations of the scaled Chebyshev nodes $\tau_i^1, \tau_i^2, \ldots, \tau_i^p$ on each interval B_i^m, and use the initial approximations Φ_0, Φ_0' to evaluate the initial approximations Φ, σ at these nodes, obtaining $\Phi_{i,m}, \sigma_{i,m}$. Choose tolerance ϵ.

Comment [Use Algorithm A to compute Newton iterates Φ_k, obtaining the solution Φ of equation (5).]

repeat

(1) Set $\hat{\Phi} = \Phi, \hat{\sigma} = \sigma$.

(2) Evaluate the functions $\hat{\Omega}, \hat{g}$ at each of the scaled Chebyshev nodes $\tau_i^1, \tau_i^2, \ldots, \tau_i^p$ on each interval B_i^m, obtaining $\hat{\Omega}_{(i,m)}, \hat{g}_{(i,m)}$.

(3) Apply Algorithm A to the discretized form of (50), obtaining $\hat{\delta}$.

(4) Set $\sigma = \hat{\sigma} + \hat{\delta}$.

(5) Compute the solution Φ of equation (49) from the values of σ, by using composite Chebyshev quadrature (see Step 6 of Algorithm A and Remark 4.5 above).

until $\|\delta_0\|_2 / \|\sigma_0\|_2 \leq \epsilon$

V. Numerical Results

FORTRAN programs have been written implementing both algorithms described in the preceding section. In this section, we discuss several details of our implementation, and demonstrate the performance of the scheme with numerical examples.

The following technical details of our implementation appear to be worth mentioning.

1. The algorithms described in the preceding section require that the number M of elementary subintervals on the interval $[a, c]$ be a power of 2. Clearly, this is not an essential limitation and it can be removed by simple bookkeeping changes. In the version of the algorithms used for numerical experiments, these changes were made.

2. Algorithm A depends for its stability on the equations (139), (140) having unique solutions for all subintervals B_i^l ($l = 0, 1, \ldots, M$, $i = 1, \ldots, 2^l$), while Algorithm B depends on (151), (152) having unique solutions for all subintervals B_i^l and for all Newton iterates k. It is easy to construct examples for which these conditions are violated, even though equation (57) or equation (32) has a unique solution. In such cases, a different subdivision of the interval $[a, c]$ can be attempted, such that none of the subintervals B_i^k of the new subdivision coincides with an interval of the original one. This procedure can be viewed as a form of pivoting, and it is easy to show that it is always possible to make it work. It has not been implemented at this point, and we have not so far encountered a need for it.

3. We have, however, implemented a crude scheme for detecting high condition numbers in the algorithms. These can occur in two places: in the solution of the linear systems on each of the finest level subintervals (Step 1 of Algorithm A), and while computing coefficients Δ_1, Δ_2 defined by (90), (91) used when merging solutions on two consecutive subintervals (Step 3 of Algorithm A). In both cases, the condition number of the system being solved is estimated in the process of solution (we use a standard LINPACK routine), and the largest of these is returned to the user. When an extremely large condition number is detected by the LINPACK routine, the resulting solution of the original ODE should be viewed as suspect. It is easy to show that when the differential operator is positive definite, this cannot happen. A more complete treatment of this subject requires further study.

4. In the upward sweep (Step 3) of Algorithm A, we evaluate the matrices $\alpha_L^{i,l}, \alpha_R^{i,l}$ for all intervals $B_{i,l}$ and use these matrices to evaluate the vectors $\delta_L^{i,l}, \delta_R^{i,l}$, the vectors $\lambda^{i,l}$, and, finally, the solution σ of the integral equation (57). But the matrices $\alpha_L^{i,l}, \alpha_R^{i,l}$ do not depend on the right-hand side f of equation (57), and it is easy to see that their evaluation accounts for more than 90% of the work. Therefore, whenever the equation (57) has to be solved with multiple right-hand sides, we can precompute the matrices $\alpha_L^{i,l}, \alpha_R^{i,l}$ and store them, saving 90% of the cost of the evaluation of subsequent solutions.

The algorithms of this paper have been applied to a variety of problems. Seven experiments are described below, and their results are summarized in Tables 1-18.

Tables 1-13 are associated with examples for which analytic solutions are available. In each of these tables, the first column contains the total number N of nodes in the discretization of the interval $[a, c]$. The second column contains the relative L^2 error of the numerical solution as compared with the analytically obtained one at 5000 equispaced points within the interval $[a, c]$, where Chebyshev interpolation has been used to evaluate the numerical solution at each of the 5000 points. The third column contains the maximum absolute error obtained at any of the 5000 points. Columns four and five contain the same information for the derivative of the solution (i.e. its relative L^2 and absolute L^∞ errors respectively). The sixth column contains the CPU time required to solve the problem, excluding the time used to evaluate the solution at 5000 equispaced points, where in all cases the times are given for a SUN SPARCstation 1 computer. Tables 11-13, associated with a nonlinear example, have in addition a seventh column which contains the number of Newton steps taken before the stopping criterion (156) has been satisfied, with $\epsilon = 10^{-10}$.

Tables 14-18 are associated with examples for which we did not have analytic solutions. In these examples, we compare each numerical solution with p Chebyshev nodes and n subintervals against the solution with p Chebyshev nodes and $2 \cdot n$ subintervals. In each of these tables, the first column contains the total number N of nodes in the discretization of the interval $[a, c]$. The second column contains the relative L^2 error of the numerical solution as compared with the numerical solution with twice the number of subintervals, where the comparison is made at each of 5000 equispaced points in the interval $[a, c]$, and where Chebyshev interpolation has been used to evaluate the numerical solution at each of the 5000 points. The third column contains the maximum absolute error obtained at any of the 5000 points. Columns four and five contain the relative L^2 and absolute L^∞ errors, respectively, for the derivative of the solution. The sixth column contains the SPARCstation CPU time required to solve the problem, excluding the time used to evaluate the solution at 5000 equispace points. Tables 16-18, associated with a nonlinear example, have in addition a seventh column which contains the number of Newton steps taken before the stopping criterion (156) has been satisfied, with $\epsilon = 10^{-10}$.

Remark 5.1 In Examples 3-4 below, we solve boundary value problems of order 2; in Example 5, we solve a problem of order 4; and in Example 7 we solve a system consisting of four equations of order 2 and two equations of order 4. In all these cases, the problems were reduced to canonical first order systems (see, for example, [4]), with the latter solved by means of algorithms A or B of the preceding section, as appropriate.

n	$E^2(\Phi)$	$E^\infty(\Phi)$	$E^2(\Phi')$	$E^\infty(\Phi')$	t (sec.)
16	0.962×10^{0}	0.685×10^{1}	0.261×10^{1}	0.673×10^{4}	0.150×10^{0}
32	0.244×10^{1}	0.216×10^{2}	0.108×10^{2}	0.169×10^{5}	0.300×10^{0}
64	0.700×10^{-1}	0.272×10^{1}	0.200×10^{1}	0.261×10^{4}	0.560×10^{0}
128	0.255×10^{-1}	0.125×10^{1}	0.754×10^{0}	0.125×10^{4}	0.108×10^{1}
256	0.667×10^{-2}	0.342×10^{0}	0.198×10^{0}	0.342×10^{3}	0.214×10^{1}
512	0.805×10^{-3}	0.536×10^{-1}	0.238×10^{-1}	0.536×10^{2}	0.428×10^{1}
1024	0.330×10^{-4}	0.291×10^{-2}	0.977×10^{-3}	0.291×10^{1}	0.846×10^{1}
2048	0.482×10^{-6}	0.529×10^{-4}	0.143×10^{-4}	0.529×10^{-1}	0.168×10^{2}
4096	0.316×10^{-8}	0.476×10^{-6}	0.935×10^{-7}	0.476×10^{-3}	0.337×10^{2}
8192	0.112×10^{-10}	0.170×10^{-8}	0.331×10^{-9}	0.170×10^{-5}	0.670×10^{2}
16384	0.115×10^{-11}	0.113×10^{-10}	0.264×10^{-11}	0.140×10^{-7}	0.137×10^{3}

Table 1: Numerical results for Example 1, $p = 8$.

Example 1 This example is taken from [2], where it is introduced as a stiff problem. The equation to be solved is given by the formulae

$$\phi_1'(x) - 998 \cdot \phi_1(x) - 1998 \cdot \phi_2(x) = 2 \cdot x, \tag{157}$$

$$\phi_2'(x) + 999 \cdot \phi_1(x) + 1999 \cdot \phi_2(x) = x, \tag{158}$$

subject to the boundary conditions

$$\phi_1(0) = 1, \tag{159}$$

$$\phi_2(1) = -6 \cdot e^{-1} + 5 \cdot e^{-1000} + .004 \cdot (.999 + .001 \cdot e^{-1000}). \tag{160}$$

We use the results of Theorem 2.3 to reduce the first order system (157) - (160) to one subject to homogeneous boundary conditions

$$\phi_1(0) = 0, \tag{161}$$

$$\phi_2(1) = 0. \tag{162}$$

We apply Algorithm A to this system using equispaced subintervals, with the number of Chebyshev nodes $p = 8, 16, 24$. For this experiment, the background Green's function is chosen to correspond to the equation

$$\Phi'(x) = 0, \tag{163}$$

subject to boundary conditions (161)-(162). The results of this experiment are presented in Tables 1-3.

Example 2 We solve the problem (157) - (160) defined in Example 1, but using an alternate division of the subintervals. Since the solution of this problem has a fairly sharp boundary

n	$E^2(\Phi)$	$E^\infty(\Phi)$	$E^2(\Phi')$	$E^\infty(\Phi')$	t (sec.)
16	0.567×10^{0}	0.436×10^{1}	0.916×10^{0}	0.397×10^{4}	0.310×10^{0}
32	0.251×10^{1}	0.192×10^{2}	0.673×10^{1}	0.175×10^{5}	0.540×10^{0}
64	0.340×10^{-1}	0.139×10^{1}	0.859×10^{0}	0.129×10^{4}	0.950×10^{0}
128	0.744×10^{-2}	0.344×10^{0}	0.220×10^{0}	0.344×10^{3}	0.179×10^{1}
256	0.798×10^{-3}	0.389×10^{-1}	0.236×10^{-1}	0.389×10^{2}	0.345×10^{1}
512	0.164×10^{-4}	0.930×10^{-2}	0.486×10^{-3}	0.930×10^{0}	0.686×10^{1}
1024	0.364×10^{-7}	0.243×10^{-5}	0.108×10^{-5}	0.243×10^{-2}	0.137×10^{2}
2048	0.992×10^{-11}	0.817×10^{-9}	0.294×10^{-9}	0.817×10^{-6}	0.274×10^{2}
4096	0.942×10^{-13}	0.776×10^{-12}	0.146×10^{-12}	0.753×10^{-9}	0.532×10^{2}
8192	0.214×10^{-12}	0.164×10^{-11}	0.328×10^{-12}	0.164×10^{-8}	0.107×10^{3}

Table 2: Numerical results for Example 1, $p = 16$.

n	$E^2(\Phi)$	$E^\infty(\Phi)$	$E^2(\Phi')$	$E^\infty(\Phi')$	t (sec.)
24	0.511×10^{0}	0.373×10^{1}	0.752×10^{0}	0.358×10^{4}	0.690×10^{0}
48	0.251×10^{0}	0.244×10^{1}	0.112×10^{1}	0.200×10^{4}	0.116×10^{1}
96	0.591×10^{-2}	0.240×10^{0}	0.175×10^{0}	0.240×10^{3}	0.209×10^{1}
192	0.496×10^{-3}	0.214×10^{-1}	0.147×10^{-1}	0.214×10^{2}	0.387×10^{1}
384	0.546×10^{-5}	0.258×10^{-3}	0.162×10^{-3}	0.258×10^{0}	0.765×10^{1}
768	0.274×10^{-8}	0.129×10^{-6}	0.811×10^{-7}	0.129×10^{-3}	0.147×10^{2}
1536	0.105×10^{-12}	0.335×10^{-11}	0.142×10^{-11}	0.256×10^{-8}	0.295×10^{2}
3072	0.663×10^{-13}	0.624×10^{-12}	0.125×10^{-12}	0.621×10^{-9}	0.578×10^{2}

Table 3: Numerical results for Example 1, $p = 24$.

n	$E^2(\Phi)$	$E^\infty(\Phi)$	$E^2(\Phi')$	$E^\infty(\Phi')$	t (sec.)
16	0.567×10^{0}	0.436×10^{1}	0.916×10^{0}	0.397×10^{4}	0.310×10^{0}
32	0.251×10^{1}	0.192×10^{2}	0.673×10^{1}	0.175×10^{5}	0.540×10^{0}
64	0.790×10^{-2}	0.360×10^{0}	0.222×10^{0}	0.345×10^{3}	0.950×10^{0}
128	0.992×10^{-11}	0.818×10^{-9}	0.294×10^{-9}	0.816×10^{-6}	0.177×10^{1}
256	0.244×10^{-12}	0.261×10^{-11}	0.525×10^{-12}	0.228×10^{-8}	0.352×10^{1}
512	0.243×10^{-12}	0.261×10^{-11}	0.525×10^{-12}	0.229×10^{-8}	0.681×10^{1}

Table 4: Numerical results for Example 2, $p = 16$.

n	$E^2(\Phi)$	$E^\infty(\Phi)$	$E^2(\Phi')$	$E^\infty(\Phi')$	t (sec.)
24	0.511×10^{0}	0.373×10^{1}	0.752×10^{0}	0.358×10^{4}	0.710×10^{0}
48	0.251×10^{0}	0.244×10^{1}	0.112×10^{1}	0.200×10^{4}	0.117×10^{1}
96	0.496×10^{-3}	0.214×10^{-1}	0.147×10^{-1}	0.214×10^{2}	0.209×10^{1}
192	0.293×10^{-12}	0.227×10^{-11}	0.507×10^{-12}	0.229×10^{-8}	0.387×10^{1}
384	0.294×10^{-12}	0.227×10^{-11}	0.509×10^{-12}	0.230×10^{-8}	0.758×10^{1}

Table 5: Numerical results for Example 2, $p = 24$.

layer near the left end of the interval $[0, 1]$, we construct the intervals $B_i^m = [b_i, b_{i+1}]$ via the formula

$$\begin{aligned} b_1 &= 0\,, \\ b_i &= \left(\frac{1}{2}\right)^{M+1-i} \qquad \text{for } i = 2, \ldots, M+1\,, \end{aligned} \tag{164}$$

so that they become progressively smaller near the left end of the interval $[0, 1]$. As in Example 1, we reduce the problem (157)- (160) to a first order system subject to homogeneous boundary conditions (161)-(162). Algorithm A has been applied to this problem using the Green's function corresponding to the equation (163) subject to boundary conditions (161)-(162), and with the number of Chebyshev nodes $p = 16$ and 24. The results of this experiment appear in Tables 4-5, and are most satisfactory.

Example 3 This example is taken from [15], where it is described as a reasonably difficult one due to the presence of rapidly growing solutions of the corresponding homogeneous equation. The equation to be solved is

$$\phi'' + 400 \cdot \phi = -400 \cdot \cos^2(\pi \cdot x) - 2 \cdot \pi^2 \cdot \cos(2 \cdot \pi \cdot x), \tag{165}$$

subject to the boundary conditions

$$\phi(0) = \phi(1) = 0. \tag{166}$$

n	$E^2(\Phi)$	$E^\infty(\Phi)$	$E^2(\Phi')$	$E^\infty(\Phi')$	t (sec.)
8	0.236×10^{0}	0.457×10^{1}	0.210×10^{0}	0.102×10^{3}	0.110×10^{0}
16	0.722×10^{-2}	0.114×10^{0}	0.733×10^{-2}	0.381×10^{1}	0.190×10^{0}
32	0.245×10^{-3}	0.331×10^{-2}	0.258×10^{-3}	0.970×10^{-1}	0.380×10^{0}
64	0.220×10^{-5}	0.410×10^{-4}	0.289×10^{-5}	0.112×10^{-2}	0.720×10^{0}
128	0.988×10^{-8}	0.255×10^{-6}	0.185×10^{-7}	0.788×10^{-5}	0.139×10^{1}
256	0.365×10^{-10}	0.115×10^{-8}	0.850×10^{-10}	0.417×10^{-7}	0.280×10^{1}
512	0.137×10^{-12}	0.464×10^{-11}	0.346×10^{-12}	0.192×10^{-7}	0.551×10^{1}
1024	0.403×10^{-14}	0.142×10^{-12}	0.755×10^{-14}	0.728×10^{-11}	0.109×10^{2}
2048	0.124×10^{-13}	0.188×10^{-12}	0.641×10^{-14}	0.381×10^{-11}	0.222×10^{2}

Table 6: Numerical results for Example 3, $p = 8$.

n	$E^2(\Phi)$	$E^\infty(\Phi)$	$E^2(\Phi')$	$E^\infty(\Phi')$	t (sec.)
16	0.125×10^{-3}	0.255×10^{-2}	0.119×10^{-3}	0.524×10^{-1}	0.360×10^{0}
32	0.290×10^{-7}	0.587×10^{-6}	0.326×10^{-7}	0.151×10^{-4}	0.650×10^{0}
64	0.555×10^{-11}	0.867×10^{-10}	0.568×10^{-11}	0.207×10^{-8}	0.119×10^{1}
128	0.217×10^{-14}	0.391×10^{-13}	0.204×10^{-14}	0.131×10^{-11}	0.231×10^{1}
256	0.262×10^{-14}	0.429×10^{-13}	0.315×10^{-14}	0.153×10^{-11}	0.454×10^{1}

Table 7: Numerical results for Example 3, $p = 16$.

We reduce the problem (165), (166) to a first order system, and apply Algorithm A to this system using equispaced subintervals, with the number of Chebyshev nodes $p = 8, 16$ and 24. For this experiment, the background Green's function is chosen to correspond to the equation

$$\Phi'(x) + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \cdot \Phi(x) = 0. \tag{167}$$

subject to boundary conditions (166). The results of this experiment are presented in Tables 6-8.

Example 4 We solve the problem (165), (166) defined in Example 3, but we use the background Green's function corresponding to the equation

$$\Phi'(x) = 0, \tag{168}$$

subject to boundary conditions (166). We reduce the problem (165), (166) to a first order system, and then use the results of Theorem 2.9 to express Φ by the formula

$$\Phi(x) = \Psi(x) \cdot \Gamma(x),$$

n	$E^2(\Phi)$	$E^\infty(\Phi)$	$E^2(\Phi')$	$E^\infty(\Phi')$	t (sec.)
24	0.164×10^{-8}	0.336×10^{-7}	0.156×10^{-8}	0.676×10^{-6}	0.760×10^{0}
48	0.338×10^{-14}	0.568×10^{-13}	0.345×10^{-14}	0.188×10^{-11}	0.136×10^{1}
96	0.264×10^{-14}	0.853×10^{-13}	0.273×10^{-14}	0.261×10^{-11}	0.257×10^{1}

Table 8: Numerical results for Example 3, $p = 24$.

with $\Psi : [0,1] \rightarrow \mathbf{L}(\mathbf{R}^{2\times 2})$ given by the formula

$$\Psi(x) = \begin{pmatrix} 1-x & x \\ -1 & 1 \end{pmatrix}, \tag{169}$$

and $\Gamma : [a,c] \rightarrow \mathbf{R}^n$ the solution to the equation

$$\Gamma'(x) + \Psi^{-1}(x) \cdot (\Psi'(x) + p(x) \cdot \Psi(x)) \cdot \Gamma(x) = \Psi^{-1}(x) \cdot f(x), \tag{170}$$

subject to boundary conditions

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \Psi(a) \cdot \Gamma(a) + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \Psi(c) \cdot \Gamma(c) = 0, \tag{171}$$

with the matrix valued function p defined by the formula

$$p(x) = \begin{pmatrix} 0 & -1 \\ 400 & 0 \end{pmatrix}, \tag{172}$$

and the vector valued function f defined via the formula

$$f(x) = \begin{pmatrix} 0 \\ -400 \cdot \cos^2(\pi \cdot x) - 2 \cdot \pi^2 \cdot \cos(2 \cdot \pi \cdot x) \end{pmatrix}. \tag{173}$$

We apply Algorithm A to this problem using equispaced subintervals, with the number of Chebyshev nodes $p = 16$ and 24. The results of this experiment are presented in Tables 9-10.

Example 5 We consider a system of Jacobian elliptic functions $sn, cn, dn : [0, 10 \cdot K] \rightarrow \mathbf{R}$ (see, for example, [1]) which are solutions to the equations

$$sn'(x) = cn(x) \cdot dn(x), \tag{174}$$
$$cn'(x) = -sn(x) \cdot dn(x), \tag{175}$$
$$dn'(x) = -m \cdot sn(x) \cdot cn(x), \tag{176}$$

with $m = \frac{1}{2}$ in our experiments, subject to the boundary conditions

$$sn(0) = 0, \tag{177}$$
$$cn(0) = 1, \tag{178}$$
$$dn(40 \cdot K) = 1, \tag{179}$$

n	$E^2(\Phi)$	$E^\infty(\Phi)$	$E^2(\Phi')$	$E^\infty(\Phi')$	t (sec.)
16	0.946×10^{-4}	0.193×10^{-2}	0.887×10^{-4}	0.412×10^{-1}	0.320×10^{0}
32	0.281×10^{-7}	0.567×10^{-6}	0.307×10^{-7}	0.145×10^{-4}	0.530×10^{0}
64	0.517×10^{-11}	0.833×10^{-10}	0.527×10^{-11}	0.198×10^{-8}	0.960×10^{0}
128	0.106×10^{-14}	0.355×10^{-13}	0.139×10^{-14}	0.796×10^{-12}	0.176×10^{1}
256	0.977×10^{-15}	0.391×10^{-13}	0.140×10^{-14}	0.853×10^{-12}	0.347×10^{1}

Table 9: Numerical results for Example 4, $p = 16$.

n	$E^2(\Phi)$	$E^\infty(\Phi)$	$E^2(\Phi')$	$E^\infty(\Phi')$	t (sec.)
24	0.113×10^{-8}	0.232×10^{-7}	0.106×10^{-8}	0.484×10^{-6}	0.700×10^{0}
48	0.470×10^{-14}	0.711×10^{-13}	0.371×10^{-14}	0.176×10^{-11}	0.116×10^{1}
96	0.180×10^{-14}	0.391×10^{-13}	0.187×10^{-14}	0.909×10^{-12}	0.207×10^{1}

Table 10: Numerical results for Example 4, $p = 24$.

with K given by the expression

$$K = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - m \cdot \sin^2 \theta}}. \tag{180}$$

We use for an initial guess the solution to (176), (179) for $m = 0$, which is defined by the formulae

$$sn(x) = \sin\left(\frac{\pi}{2 \cdot K} \cdot x\right), \tag{181}$$

$$cn(x) = \cos\left(\frac{\pi}{2 \cdot K} \cdot x\right), \tag{182}$$

$$dn(x) = 1. \tag{183}$$

We reduce the problem (165), (166) to a first order system, and then use the results of Theorem 2.4 to reduce this system to one subject to the homogeneous boundary conditions

$$sn(0) = 0, \tag{184}$$

$$cn(0) = 0, \tag{185}$$

$$dn(40 \cdot K) = 0. \tag{186}$$

We then apply Algorithm B to this system using equispaced subintervals, with the number of Chebyshev nodes $p = 8, 16$ and 32. For this experiment, the background Green's function is chosen to correspond to the equation

$$\Phi'(x) = 0,$$

n	$E^2(\Phi)$	$E^\infty(\Phi)$	$E^2(\Phi')$	$E^\infty(\Phi')$	t (sec.)	Steps
128	0.211×10^{-1}	0.551×10^{-1}	0.333×10^{-1}	0.620×10^{-1}	0.167×10^{2}	9
256	0.158×10^{-2}	0.416×10^{-2}	0.228×10^{-2}	0.430×10^{-2}	0.258×10^{2}	7
512	0.849×10^{-5}	0.226×10^{-4}	0.123×10^{-4}	0.240×10^{-4}	0.439×10^{2}	6
1024	0.106×10^{-7}	0.330×10^{-7}	0.180×10^{-7}	0.460×10^{-7}	0.883×10^{2}	6
2048	0.313×10^{-10}	0.984×10^{-10}	0.599×10^{-10}	0.150×10^{-9}	0.175×10^{3}	6
4096	0.147×10^{-12}	0.469×10^{-12}	0.264×10^{-12}	0.674×10^{-12}	0.348×10^{3}	6

Table 11: Numerical results for Example 5, $p = 8$.

n	$E^2(\Phi)$	$E^\infty(\Phi)$	$E^2(\Phi')$	$E^\infty(\Phi')$	t (sec.)	Steps
64	0.162×10^{0}	0.505×10^{0}	0.244×10^{0}	0.546×10^{0}	0.177×10^{2}	10
128	0.422×10^{-1}	0.108×10^{0}	0.612×10^{-1}	0.113×10^{0}	0.246×10^{2}	7
256	0.181×10^{-3}	0.476×10^{-3}	0.268×10^{-3}	0.491×10^{-3}	0.409×10^{2}	6
512	0.441×10^{-7}	0.120×10^{-6}	0.672×10^{-7}	0.159×10^{-6}	0.851×10^{2}	6
1024	0.425×10^{-12}	0.125×10^{-11}	0.115×10^{-11}	0.293×10^{-11}	0.164×10^{3}	6
2048	0.115×10^{-12}	0.317×10^{-12}	0.164×10^{-12}	0.317×10^{-12}	0.324×10^{3}	6
4096	0.569×10^{-13}	0.152×10^{-12}	0.814×10^{-13}	0.154×10^{-12}	0.653×10^{3}	6

Table 12: Numerical results for Example 5, $p = 16$.

subject to boundary conditions (184) - (186). The results of this experiment are presented in Tables 11-13.

Example 6 This example is taken from [14]. Its purpose is to demonstrate the performance of the method when the equation to be solved contains fourth order derivatives. The deflection of a beam under a uniform load q, with the beam built in at the left end ($x = 0$) and simply supported at the right end, is given by the formula

$$y''''(x) + \frac{k}{E \cdot I} \cdot y(x) = \frac{q}{E \cdot I}, \tag{187}$$

subject to the boundary conditions

$$y(0) = y'(0) = y(L) = y''(L) = 0, \tag{188}$$

with k the force per unit deflection per unit length of beam, and $E \cdot I$ the flexural rigidity of the beam. The values of the constants used are

$$L = 1.2 \times 10^2 \text{ in.}, \tag{189}$$

$$k = 2.604 \times 10^3 \text{ psi}, \tag{190}$$

n	$E^2(\Phi)$	$E^\infty(\Phi)$	$E^2(\Phi')$	$E^\infty(\Phi')$	t (sec.)	Steps
256	0.148×10^{-3}	0.388×10^{-3}	0.215×10^{-3}	0.411×10^{-3}	0.163×10^{3}	6
512	0.124×10^{-9}	0.395×10^{-9}	0.961×10^{-9}	0.186×10^{-8}	0.322×10^{3}	6
1024	0.450×10^{-12}	0.119×10^{-11}	0.643×10^{-12}	0.119×10^{-11}	0.632×10^{3}	6
2048	0.266×10^{-12}	0.703×10^{-12}	0.381×10^{-12}	0.703×10^{-12}	0.126×10^{4}	6

Table 13: Numerical results for Example 5, $p = 32$.

$$q = 4.34 \times 10^4 \text{ lbs/in.}, \tag{191}$$
$$E = 3.0 \times 10^7 \text{ psi}, \tag{192}$$
$$I = 3.0 \times 10^3 \text{ in.}^4, \tag{193}$$

(see [14], p. 174). The L^2 norm of y, the solution to (188), is approximately 10^6 times larger than the L^2 norm of y''''. Combined with the high number of derivatives in (187), this tends to present difficulties for finite difference methods. We reduce the problem (187), (188) to a first order system, and then use the results of Theorem 2.9 to express Φ by the formula

$$\Phi(x) = \Psi(x) \cdot \Gamma(x),$$

with $\Psi(x) : [0, L] \to \mathbf{L}(\mathbf{R}^{4\times 4})$ given by the formula

$$\Psi(x) = \begin{pmatrix} \frac{L-x}{L} & 0 & \frac{x}{L} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}. \tag{194}$$

and $\Gamma : [a, c] \to \mathbf{R}^n$ the solution to the equation

$$\Gamma'(x) + \Psi^{-1}(x) \cdot (\Psi'(x) + p(x) \cdot \Psi(x)) \cdot \Gamma(x) = \Psi^{-1}(x) \cdot f(x), \tag{195}$$

subject to boundary conditions

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \Psi(a) \cdot \Gamma(a) + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \Psi(c) \cdot \Gamma(c) = 0, \tag{196}$$

with the matrix valued function p defined by the formula

$$p(x) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ \frac{k}{E \cdot I} & 0 & 0 & 0 \end{pmatrix}, \tag{197}$$

n	$E^2(\Phi)$	$E^\infty(\Phi)$	$E^2(\Phi')$	$E^\infty(\Phi')$	t (sec.)
4	0.115×10^{0}	0.672×10^{-1}	0.277×10^{-1}	0.433×10^{-3}	0.110×10^{0}
8	0.755×10^{-2}	0.506×10^{-2}	0.215×10^{-2}	0.426×10^{-4}	0.220×10^{0}
16	0.474×10^{-3}	0.369×10^{-3}	0.141×10^{-3}	0.317×10^{-5}	0.400×10^{0}
32	0.269×10^{-4}	0.249×10^{-4}	0.891×10^{-5}	0.212×10^{-6}	0.780×10^{0}
64	0.185×10^{-5}	0.162×10^{-5}	0.554×10^{-6}	0.137×10^{-7}	0.153×10^{1}
128	0.117×10^{-6}	0.103×10^{-6}	0.338×10^{-7}	0.866×10^{-9}	0.303×10^{1}
256	0.891×10^{-8}	0.652×10^{-8}	0.223×10^{-8}	0.539×10^{-10}	0.603×10^{1}
512	0.306×10^{-8}	0.170×10^{-8}	0.338×10^{-8}	0.416×10^{-10}	0.120×10^{2}

Table 14: Numerical results for Example 6, $p = 4$.

n	$E^2(\Phi)$	$E^\infty(\Phi)$	$E^2(\Phi')$	$E^\infty(\Phi')$	t (sec.)
8	0.130×10^{-5}	0.816×10^{-6}	0.688×10^{-6}	0.139×10^{-7}	0.280×10^{0}
16	0.792×10^{-8}	0.510×10^{-8}	0.101×10^{-7}	0.149×10^{-9}	0.530×10^{0}
32	0.465×10^{-8}	0.235×10^{-8}	0.468×10^{-8}	0.565×10^{-10}	0.105×10^{1}
64	0.916×10^{-8}	0.463×10^{-8}	0.927×10^{-8}	0.111×10^{-9}	0.201×10^{1}

Table 15: Numerical results for Example 6, $p = 8$.

and the vector valued function f defined via the formula

$$f(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{q}{E \cdot I} \end{pmatrix}. \tag{198}$$

We apply Algorithm A to this problem using equispace subintervals, with the number of Chebyshev nodes $p = 4$ and 8. For this experiment the background Green's function is chosen to correspond to the equation

$$\Phi'(x) = 0,$$

subject to boundary conditions (188). The results of this experiment are presented in Tables 14-15.

Example 7 This example is taken from [13], and was developed in cooperation with the author of [13]. The problem is a system of six nonlinear equations with inhomogeneous boundary conditions, and is described in detail in Appendix A. Since 2 of the ODEs are fourth order, and 4 are second order, the problem reduces to a system of 16 first order nonlinear equations. The 2 fourth order ODEs have in their fourth derivatives a boundary layer of order 10^5 at the left end of the interval.

n	$E^2(\Phi)$	$E^\infty(\Phi)$	$E^2(\Phi')$	$E^\infty(\Phi')$	t (sec.)	Steps
80	0.697×10^{-1}	0.516×10^{2}	0.118×10^{0}	0.415×10^{4}	0.249×10^{3}	6
160	0.854×10^{-2}	0.773×10^{1}	0.167×10^{-1}	0.790×10^{3}	0.542×10^{3}	6
320	0.795×10^{-3}	0.952×10^{0}	0.177×10^{-2}	0.111×10^{3}	0.105×10^{4}	6
640	0.589×10^{-4}	0.917×10^{-1}	0.143×10^{-3}	0.114×10^{2}	0.205×10^{4}	6

Table 16: Numerical results for Example 7, $p = 4$.

n	$E^2(\Phi)$	$E^\infty(\Phi)$	$E^2(\Phi')$	$E^\infty(\Phi')$	t (sec.)	Steps
80	0.814×10^{-2}	0.686×10^{1}	0.151×10^{-1}	0.667×10^{3}	0.633×10^{3}	6
160	0.409×10^{-2}	0.380×10^{1}	0.871×10^{-3}	0.454×10^{2}	0.107×10^{4}	6
320	0.884×10^{-5}	0.977×10^{-2}	0.238×10^{-4}	0.140×10^{1}	0.206×10^{4}	6
640	0.930×10^{-7}	0.130×10^{-3}	0.312×10^{-6}	0.213×10^{-1}	0.396×10^{4}	6

Table 17: Numerical results for Example 7, $p = 8$.

We use for an initial guess the solution to the problem for $n = 40, p = 4$. We apply Algorithm B to this system using equispaced subintervals, with the number of Chebyshev nodes $p = 4, 8$, and 16. The results of this experiment are presented in Tables 16-18.

The following observations can be made from Tables 1 - 18, and are corroborated by our more extensive experiments.

1. The practical convergence rate of the method is consistent with the theoretical one. For larger p, the exact numerical verification of the order of convergence tends to be difficult, since the precision of calculations is exhausted before the behavior of the scheme becomes asymptotic. However, this is often encountered when dealing with rapidly convergent algorithms.

2. For small-scale problems and large p, the algorithm produces essentially exact results with a small number of nodes. For large-scale problems, double precision accuracy is achieved at approximately 20 nodes per wavelength with $p = 20$, at 12 nodes per wavelength with $p = 24$.

n	$E^2(\Phi)$	$E^\infty(\Phi)$	$E^2(\Phi')$	$E^\infty(\Phi')$	t (sec.)	Steps
80	0.301×10^{-3}	0.269×10^{0}	0.673×10^{-3}	0.338×10^{2}	0.155×10^{4}	6
160	0.419×10^{-5}	0.393×10^{-2}	0.106×10^{-4}	0.557×10^{0}	0.291×10^{4}	6
320	0.114×10^{-7}	0.115×10^{-4}	0.378×10^{-7}	0.199×10^{-2}	0.565×10^{4}	6
640	0.624×10^{-11}	0.728×10^{-8}	0.287×10^{-10}	0.158×10^{-5}	0.112×10^{5}	6

Table 18: Numerical results for Example 7, $p = 16$.

and at 10 nodes per wavelength with $p = 32$. The optimal timings are achieved at p between 24 and 32 (provided that about 10 - 12 digits of accuracy are desired).

3. The condition number of a Nyström discretization of a second kind integral equation is asymptotically bounded, and our results reflect this fact. The relatively poor accuracy (8 - 11 digits) obtained in Examples 6 and 7 is due to the ill-conditioning of the original ODE, as opposed to that of the numerical scheme used.

4. The algorithm is completely indifferent to the stiffness near the left end of the interval $[0, 1]$ of equations (157), (158) in Examples 1-2.

5. It is easy to use the algorithm in an adaptive manner, as demonstrated in Example 2. However, a fully adaptive version of the scheme has not been implemented. The intervals B_i^m in Example 2 were provided by the calling program (as opposed to having been constructed by the algorithm itself).

6. The numerical advantages of one background Green's function over another tend to be minor, as indicated in Examples 3-4. However, using the Green's function given in Lemma 2.2 does result in a slightly faster algorithm. This is because this Green's function is constant in each of the intervals $(x \leq t), (x \geq t)$, which provides in Step 2 of Algorithm A faster evaluation of the matrices $\alpha_L^{i,m}, \alpha_R^{i,m}$ given by equations (142)-(143) and vectors $\delta_L^{i,m}, \delta_R^{i,m}$ given by equations (144), (145), and provides in Step 6 a faster evaluation of the solution Φ of equation (27).

7. The algorithm can solve systems of high order equations with no numerical difficulty, as demonstrated by Examples 6-7.

VI. Conclusions

An algorithm has been presented for the solution of two-point boundary value problems of ordinary differential equations. The algorithm is based on reducing the differential equation to a second kind integral equation, with the subsequent solution of the latter via a Nyström type scheme. It has CPU time requirements proportional to $N \cdot p^2 \cdot n^3$, with N the number of nodes in the discretization of the interval of definition of the equation, p the desired order of convergence of the scheme, and n the number of equations in the first order system. The method does not involve the solution of linear systems with large condition numbers, permits the use of schemes with extremely high orders of convergence, and is quite insensitive to boundary layers or to end-point singularities in the coefficients of the differential equation.

The algorithm has been combined with Newton's method, resulting in a scheme for the solution of boundary value problems for nonlinear ODEs. In this case, each Newton iterate is expressed as the solution of a linear second kind integral equation; the analytical and numerical advantages of integral equations are thus obtained for nonlinear boundary value problems.

Acknowledgments

The authors would like to thank Professor L. Greengard and Y. Chen for their interest and help, and D. Koch for providing Example 7 of Section 5.

Appendix A.
Numerical Solution of Square-Cell Convection with Strongly Variable Viscosity

This problem is introduced in [13] as a model for square-cell convection with strongly variable viscosity. The model is used to examine the influence of temperature-dependent and pressure-dependent viscosity on convective heat transfer and surface motion. The problem is given by the system of six coupled, nonlinear equations

$$\begin{aligned}\phi_1''''(x) &= (4\cdot R\cdot\theta_1(x)+4\cdot(2\cdot a_k^2\cdot\phi_1''(x)-a_k^4\cdot\phi_1(x)+\gamma\cdot(\phi_1''(x)\cdot\overline{T}''(x) \\ &\quad +2\cdot\phi_1'''(x)\cdot\overline{T}'(x)-2\cdot a_k^2\cdot\phi_1'(x)\cdot\overline{T}'(x)-2\cdot a_k^2\cdot\phi_1''(x)\cdot\overline{T}(x) \\ &\quad +a_k^2\cdot\overline{T}''(x)\cdot\phi_1(x)+a_k^4\cdot\phi_1(x)\cdot\overline{T}(x)))+2\cdot\gamma\cdot(\phi_2''(x)\cdot\theta_1''(x) \\ &\quad +2\cdot\phi_2'''(x)\cdot\theta_1'(x)+\phi_2''''(x)\cdot\theta_1(x)-2\cdot a_k^2\cdot\phi_2'(x)\cdot\theta_1'(x)-3\cdot a_k^2\cdot\phi_2''(x)\cdot\theta_1(x) \\ &\quad +2\cdot a_k^2\cdot\phi_2(x)\cdot\theta_1''(x)+2\cdot a_k^4\cdot\phi_2(x)\cdot\theta_1(x)-2\cdot a_k^2\cdot\phi_1'(x)\cdot\theta_2'(x) \\ &\quad -2\cdot a_k^2\cdot\phi_1''(x)\cdot\theta_2(x))+\gamma\cdot(\psi'(x)\cdot\theta_2''(x)+2\cdot\psi''(x)\cdot\theta_2'(x)+\psi'''(x)\cdot\theta_2(x) \\ &\quad -3\cdot a_k^2\cdot\psi'(x)\cdot\theta_2(x)-4\cdot a_k^2\cdot\theta_2'(x)\cdot\psi(x)))/(4\cdot(1-\gamma\cdot\overline{T}(x))).\end{aligned} \tag{199}$$

$$\begin{aligned}\phi_2''''(x) &= (2\cdot R\cdot\theta_2(x)+2\cdot(4\cdot a_k^2\cdot\phi_2''(x)-4\cdot a_k^4\cdot\phi_2(x)+\gamma\cdot(\phi_2''(x)\cdot\overline{T}''(x) \\ &\quad +2\cdot\phi_2'''(x)\cdot\overline{T}'(x)-4\cdot a_k^2\cdot\phi_2'(x)\cdot\overline{T}'(x)-4\cdot a_k^2\cdot\phi_2''(x)\cdot\overline{T}(x) \\ &\quad +2\cdot a_k^2\cdot\overline{T}''(x)\cdot\phi_2(x)+4\cdot a_k^4\cdot\phi_2(x)\cdot\overline{T}(x)+\phi_1''(x)\cdot\theta_1''(x)+2\cdot\phi_1'''(x)\cdot\theta_1'(x) \\ &\quad +\phi_1''''(x)\cdot\theta_1(x)-4\cdot a_k^2\cdot\phi_1'(x)\cdot\theta_1'(x)-3\cdot a_k^2\cdot\phi_1''(x)\cdot\theta_1(x) \\ &\quad +a_k^2\cdot\phi_1(x)\cdot\theta_1''(x)+2\cdot a_k^2\cdot\phi_1(x)\cdot\theta_1(x)))+\gamma\cdot(-\psi''(x)\cdot\theta_1''(x) \\ &\quad -2\cdot\psi''(x)\cdot\theta_1'(x)-\psi'''(x)\cdot\theta_1(x)+4\cdot a_k^2\cdot\psi'(x)\cdot\theta_1(x)+6\cdot a_k^2\cdot\theta_1'(x)\cdot\psi(x))) \\ &\quad /(2\cdot(1-\gamma\cdot\overline{T}(x))),\end{aligned} \tag{200}$$

$$\overline{T}''(x) = a_k^2\cdot(\phi_1'(x)\cdot\theta_1(x)+\phi_1(x)\cdot\theta_1'(x)+\frac{1}{2}\cdot\phi_2'(x)\cdot\theta_2(x)+\frac{1}{2}\cdot\phi_2(x)\cdot\theta_2'(x)). \tag{201}$$

$$\begin{aligned}\theta_1''(x) &= \frac{a_k^2}{4}\cdot(4\ \ (\theta_1(x)+\overline{T}'(x)\cdot\phi_1(x))+2\cdot\theta_2(x)\cdot\phi_1'(x)+2\cdot\theta_2'(x)\cdot\phi_1(x) \\ &\quad +2\cdot\phi_2'(x)\cdot\theta_1(x)+4\cdot\theta_1'(x)\cdot\phi_2(x)-\psi(x)\cdot\theta_2(x)).\end{aligned} \tag{202}$$

$$\theta_2''(x) = a_k^2\cdot(2\cdot\theta_2(x)+2\cdot\overline{T}'(x)\cdot\phi_2(x)+2\cdot\theta_1'(x)\cdot\phi_1(x)+\psi(x)\cdot\theta_1(x)). \tag{203}$$

$$\begin{aligned}\psi''(x) &= (\gamma\cdot(\phi_1''(x)\cdot\theta_2'(x)+\phi_1'''(x)\cdot\theta_2(x)-3\cdot a_k^2\cdot\phi_1'(x)\cdot\theta_2(x)+a_k^2\cdot\phi_1(x)\cdot\theta_2'(x) \\ &\quad -\phi_2''(x)\cdot\theta_1'(x)-\phi_2'''(x)\cdot\theta_1(x)+4\cdot a_k^2\cdot\phi_2'(x)\cdot\theta_1(x)-2\cdot a_k^2\cdot\phi_2(x)\cdot\theta_1'(x)) \\ &\quad +10\cdot(5\cdot a_k^2\cdot\psi(x)+\gamma\cdot(\psi'(x)\cdot\overline{T}'(x)-5\cdot a_k^2\cdot\psi(x)\cdot\overline{T}(x))) \\ &\quad +\frac{\gamma}{2}\cdot(-4\cdot\psi'(x)\cdot\theta_2'(x)+7\cdot a_k^2\cdot\psi(x)\cdot\theta_2(x)))/(10-10\cdot\gamma\cdot\overline{T}(x)+2\cdot\gamma\cdot\theta_2(x)).\end{aligned} \tag{204}$$

subject to boundary conditions

$$\phi_1(a) = \phi_1''(a) = \phi_2(a) = \phi_2''(a) = \overline{T}(a) - \frac{1}{2} = \theta_1(a) = \theta_2(a) = \psi'(a) = 0, \quad (205)$$

$$\phi_1(c) = \phi_1''(c) = \phi_2(c) = \phi_2''(c) = \overline{T}(c) - \frac{1}{2} = \theta_1(c) = \theta_2(c) = \psi'(c) = 0, \quad (206)$$

and with $x \in [-1/2, 1/2]$. We use as an initial guess the approximations

$$\phi_1(x) = \epsilon \cdot \cos(\pi \cdot x), \quad (207)$$

$$\phi_2(x) = \epsilon^2 \cdot c_2 \cdot \sin(2 \cdot \pi \cdot x), \quad (208)$$

$$\overline{T}(x) = \epsilon^2 \cdot c_3 \cdot \sin(2 \cdot \pi \cdot x) - x, \quad (209)$$

$$\theta_1(x) = \epsilon \cdot c_1 \cdot \cos(\pi \cdot x), \quad (210)$$

$$\theta_2(x) = \epsilon^2 \cdot c_4 \cdot \sin(2 \cdot \pi \cdot x), \quad (211)$$

$$\psi(x) = \epsilon^3 \cdot \gamma \cdot (c_5 \cdot \frac{\cos(\pi \cdot x) + D_d \cdot \cosh(\sqrt{5} \cdot a_k \cdot x)}{\pi^2 + 5 \cdot a_k^2} + c_6 \cdot \frac{\cos(3 \cdot \pi \cdot x) - 3 \cdot D_d \cdot \cosh(\sqrt{5} a_k x)}{9\pi^2 + 5a_k^2}) / (5a_k^2). \quad (212)$$

The constants in this experiment are given by the formulae

$$R = 2000, \quad (213)$$

$$a_k = 2.2, \quad (214)$$

$$r_r = 10, \quad (215)$$

$$\gamma = \frac{2 \cdot (r_r - 1)}{r_r + 1}, \quad (216)$$

$$r_{00} = 8 \cdot \pi^6 \cdot a_k^4, \quad (217)$$

$$c_1 = \frac{a_k^2}{\pi^2 + a_k^2}, \quad (218)$$

$$c_2 = \frac{-\pi \cdot a_k^2 \cdot c_1 \cdot r_{00}}{2 \cdot a_k^2 \cdot r_{00} - 8 \cdot r_{00} \cdot a_k^2}, \quad (219)$$

$$c_3 = \frac{a_k^2 \cdot c_1}{4 \cdot \pi}, \quad (220)$$

$$c_4 = \frac{a_k^2 \cdot (2 \cdot c_2 + \pi \cdot c_1)}{2 \cdot r_{00} \cdot a_k^2}, \quad (221)$$

$$c_5 = \pi \cdot \frac{a_k^2}{2} \cdot (\frac{c_4}{2} \cdot (\pi^2 - 5 \cdot a_k^2) - c_1 \cdot c_2 \cdot (2 \cdot \pi^2 + 5 \cdot a_k^2)), \tag{222}$$

$$c_6 = \pi \cdot \frac{a_k^2}{2} \cdot (\frac{c_4}{2} \cdot (3 \cdot \pi^2 + a_k^2) - 3 \cdot c_1 \cdot c_2 \cdot r_{00} \cdot a_k^2), \tag{223}$$

$$D_d = \pi/(\sqrt{5} \cdot a_k \cdot \sinh(\sqrt{5} \cdot a_k/2)), \tag{224}$$

$$\epsilon = \sqrt{\frac{R - r_{00}}{\pi \cdot (c_3 + c_4/4) \cdot r_{00}}}. \tag{225}$$

In order to reduce the problem to a system of integral equations, we first solve for ϕ_1'''', ϕ_2'''' in the linear system involving $\phi_1'''', \phi_2'''', \psi'''$, obtaining a new problem which does not contain dependencies among the variables. We reduce this new problem to a first order system, and then use the results of Theorem 2.10 to express Φ by the formula

$$\Phi(x) = \Psi(x) \cdot \Gamma(x),$$

with $\Psi : [a, c] \to \mathbf{L}(\mathbf{R}^{16\times 16})$ given by the formula

$$\Psi(x) = \mathrm{diag}(\psi_A, \psi_A, \psi_A, \psi_A, \psi_A, \psi_A, \psi_A, \psi_B), \tag{226}$$

with $\psi_A, \psi_B : [-\frac{1}{2}, \frac{1}{2}] \to \mathbf{L}(\mathbf{R}^{2\times 2})$ defined by the formulae

$$\psi_A(x) = \begin{pmatrix} \frac{1}{2} - x & \frac{1}{2} + x \\ -1 & 1 \end{pmatrix}, \tag{227}$$

$$\psi_B(x) = \begin{pmatrix} -1 & 1 \\ \frac{1}{2} - x & \frac{1}{2} + x \end{pmatrix}, \tag{228}$$

and with $\Gamma : [a, c] \to \mathbf{R}^n$ the solution to an equation of the form (40), subject to boundary conditions of the form (41). The results of Theorem 2.4 are then applied to reduce this new problem with inhomogeneous boundary conditions to a problem which boundary conditions are homogeneous. For these experiments, the background Green's function is chosen to correspond to the equation

$$\Phi'(x) = 0, \tag{229}$$

subject to boundary conditions (206).

References

[1] M. Abramowitz and I. A. Stegun, ed., *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Mathematics Series 55, 1964.

[2] O. R. Asfar and A. M. Hussein, *Numerical Solution of Linear Two-Point Boundary Problems via the Fundamental-Matrix Method*, Int. Jour. for Num. Meth. in Eng. vol. 28 (1989). pp. 1205-1216.

[3] G. A. Bliss, *A Boundary Value Problem for a System of Ordinary Linear Differential Equations of the First Order*, Trans. Amer. Math. Soc. vol. 28 (1926), pp. 561-589.

[4] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, R. E. Krieger Publishing Company, 1984.

[5] R. Courant and D. Hilbert, *Methods of Mathematical Physics, Vol I*, Wiley Interscience, New York, 1953.

[6] P. J. Davis and P. Rabinowitz, *Methods of Numerical Integration*, Academic Press, Orlando, 1984.

[7] J. E. Dennis, Jr. and R. B. Schnabel, *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*, Prentice-Hall, Englewoods Cliffs, New Jersey, 1983.

[8] L. Fox and I. B. Parker, *Chebyshev Polynomials in Numerical Analysis*, Oxford University Press, London, 1968.

[9] G. H. Golub and C. F. Van Loan, *Matrix Computations*, Johns Hopkins University Press, Baltimore, 1983.

[10] D. Gottlieb and S. Orszag, *Numerical Analysis of Spectral Methods*, SIAM, Philadelphia, 1977.

[11] L. Greengard and V. Rokhlin, *On the Numerical Solution of Two-Point Boundary Value Problems*, Technical Report 692, Yale University, Department of Computer Science, 1989. to appear in *Communications in Pure and Applied Mathematics*.

[12] H. B. Keller, *Numerical Solution of Two Point Boundary Value Problems*, SIAM, Philadelphia, 1976.

[13] D. Koch, *Modal Solutions for Convection With Strongly Variable Viscosity at High Rayleigh Numbers*, in preparation.

[14] M. G. Salvadori and M. L. Baron, *Numerical Methods in Engineering*, Prentice-Hall, New Jersey, 1961.

[15] J. Stoer and R. Bulirsch, *Introduction to Numerical Analysis*, Springer-Verlag, New York, 1980.